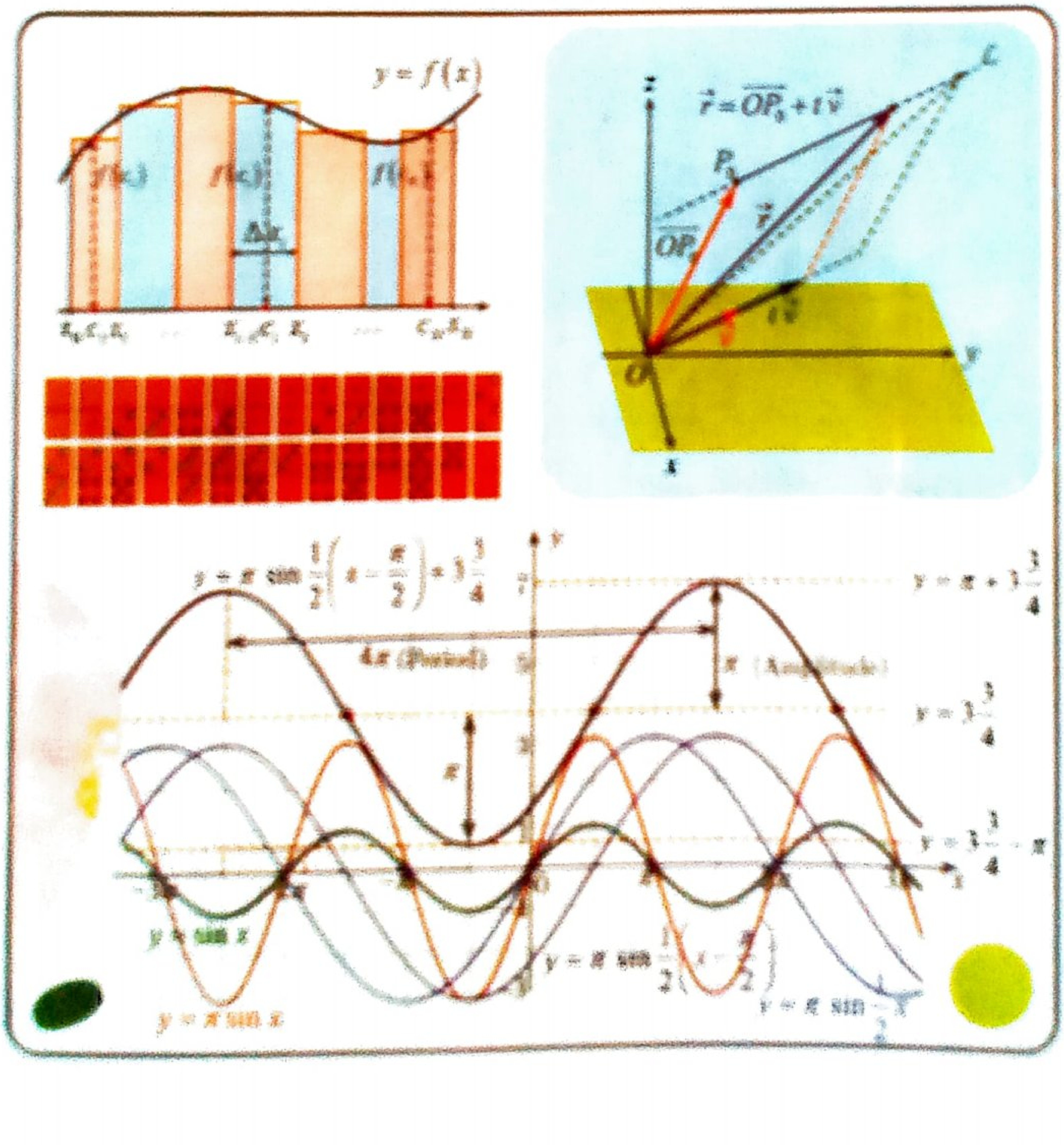
THE GOVERNMENT OF
THE REPUBLIC OF THE UNION OF MYANMAR
MINISTRY OF EDUCATION

TEXTBOOK

MATHEMATICS

GRADE 12


y = f(x)
Δx
r = OP₀ + tv
P₀
OP₀
tv
O
4π (Period)
π (Amplitude)
y = sin x
y = π sin x

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TEXTBOOK

MATHEMATICS

GRADE 12

BASIC EDUCATION CURRICULUM AND TEXTBOOK COMMITTEE

၂၀၂၃ ခုနှစ်၊ ဇန်နဝါရီလ၊ အုပ်ရေ - ၅၂၂၃၈၂ အုပ်
၂၀၂၃ - ၂၀၂၄ ပညာသင်နှစ်

အမျိုးသားသင်ရိုးညွှန်းတမ်းကော်မတီကအတည်ပြုသည်။
အခြေခံပညာသင်ရိုးညွှန်းတမ်းနှင့်
ကျောင်းသုံးစာအုပ်ကော်မတီ၏မူပိုင်ဖြစ်သည်။

အလုပ်အမိန့်အမှတ် - L ၀၃/၂၃ ဖြင့်
မြဝတီစာပေတိုက်၊ ရန်ကုန်မြို့တွင် ပုံနှိပ်သည်။

TEXTBOOK
MATHEMATICS
GRADE 12

Preface

This textbook is the Grade 12 High School Mathematics written as a new curriculum Most of the chapters develop mathematical concepts and skills through understandin fundamental ideas in Mathematics.

The aims of teaching Mathematics at the high school level are as follows: At th Basic Education High School Level, students should be able

- to gain basic mathematical knowledge and understanding,
- to acquire the necessary mathematical skills,
- to apply mathematical knowledge and skill in real-life situations, and
- to gain interest in Mathematics and appreciate the values of Mathematics.

Chapter 1 introduces the new number system. This chapter expresses the complex numbers, operations on complex numbers, trigonometric forms and roots o complex numbers.

Chapter 2 presents the principle of mathematical induction. Some mathematica statements are proved by using the principle of mathematical induction.

Chapter 3 expresses the analytic solid geometry such as line, plane and sphere.

Chapter 4 introduces the vectors in three dimensions. Scalar product, vecto product, lines and planes in three-dimensional space are illustrated.

Chapter 5 describes the concepts of permutation and combination that can appl to many areas of Mathematics, Statistics, Science and Engineering.

Chapter 6 expresses the basic properties and equations of circles and parabola Also, the definitions of conic sections, a general form of the equation of a circle, general form of the equation of a parabola, and the translation and rotation of axe are discussed.

Chapter 7 illustrates the graphs of sines, cosines, tangents and other trigonomet nctions. Translation, reflection and scaling of those graphs, inverse trigonomet nctions and their graphs, and the differentiation of trigonometric functions a scribed.

Chapter 8 introduces the graph of logarithmic and exponential functions. ranslation, reflection and scaling of those graphs, the differentiation of logarithm d exponential functions are described.

Chapter 9 starts with the derivative of several functions such as polynomia tional, exponential, logarithmic and trigonometric functions. A critical poin aximum, minimum, second derivative test and approximation are discussed.

Chapter 10 is the chapter on method of integration. Basic integratio bstitution method, integration by parts and partial fraction method are illustrate

Chapter 11 is the last chapter of this textbook. Application of integration suc the area under the curve and volume integral is illustrated.

After learning this course, students will develop and practise higher-order thinki ills: comprehension, analysis, synthesis and evaluation. They will be able articipate actively in all lessons through the 5 C's as important 21st century sk r learning:

- **Collaboration** - in the lesson students will be working in groups, to share ide with their classmates and to find the solution together
- **Communication** - students will develop verbal and non-verbal communicatio skills in group works
- **Critical thinking and problem solving** - students will be given interesti problems to solve-finding and explaining solutions, looking for correcting erro
- **Creativity and innovation** - thinking 'outside the box' is an important 21 century skill. Students will be encouraged to explore new ideas and so problems in new ways
- **Citizenship** - students will join the school community and develop fairness an conflict resolution skills.

is textbook aims at applying basic mathematical knowledge and skills in real-life udents.

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Chapter 1

Complex Numbers

In this chapter we introduce a new number system which is the extension of the real number system.

1.1 Pure Imaginary Unit i

Since we have known that $x^2 \geq 0$ for every real number x, the equation $x^2 = -4$ has no real solution. But if there is a pure imaginary unit i such that

$$\boxed{i^2 = -1}$$

with acceptance of the usual operations on real numbers and i such as

$$(2i)^2 = 4i^2 = 4(-1) = -4 \quad \text{and} \quad (-2i)^2 = 4i^2 = 4(-1) = -4$$

then $x^2 = -4$ has two solutions $2i$ and $-2i$.

If we try to solve $x^2 = -n$, for $n > 0$, like $x^2 = -4$, then $\sqrt{n}\,i$ and $-\sqrt{n}\,i$ are solutions because

$$\boxed{(\pm\sqrt{n}\,i)^2 = ni^2 = n(-1) = -n}$$

Example 1.

Solve $x^2 + 2x + 5 = 0$.

Solution

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ x^2 + 2x + 1 &= -4 \\ (x+1)^2 &= -4 \\ x + 1 &= \pm\sqrt{4}\,i \end{aligned}$$

$$x = -1 \pm 2i$$

So there are two solutions $x = -1 + 2i$ and $x = -1 - 2i$.

Example 2.

Solve $x^2 + 2x + 3 = 0$ and check your answers.

Solution

$$\begin{aligned} x^2 + 2x + 3 &= 0 \\ x^2 + 2x + 1 &= -2 \\ (x+1)^2 &= -2 \\ x + 1 &= \pm\sqrt{2}\,i \\ x &= -1 \pm \sqrt{2}\,i \end{aligned}$$

So two solutions are $x = -1 + \sqrt{2}\,i$ and $x = -1 - \sqrt{2}\,i$.

For $x = -1 + \sqrt{2}\,i$,

$$\begin{aligned} x^2 + 2x + 3 &= (-1 + \sqrt{2}\,i)^2 + 2(-1 + \sqrt{2}\,i) + 3 \\ &= 1 - 2\sqrt{2}\,i + 2i^2 - 2 + 2\sqrt{2}\,i + 3 \\ &= 1 - 2\sqrt{2}\,i - 2 - 2 + 2\sqrt{2}\,i + 3 \\ &= 0 \end{aligned}$$

For $x = -1 - \sqrt{2}\,i$,

$$\begin{aligned} x^2 + 2x + 3 &= (-1 - \sqrt{2}\,i)^2 + 2(-1 - \sqrt{2}\,i) + 3 \\ &= 1 + 2\sqrt{2}\,i + 2i^2 - 2 - 2\sqrt{2}\,i + 3 \\ &= 1 + 2\sqrt{2}\,i - 2 - 2 - 2\sqrt{2}\,i + 3 \\ &= 0 \end{aligned}$$

Exercise 1.1

1. Solve the following equations.

   (a) $x^2 - 6x + 10 = 0$
   (b) $-2x^2 + 4x - 3 = 0$
   (c) $5x^2 - 2x + 1 = 0$
   (d) $3x^2 + 7x + 5 = 0$

2. Solve the following equations and check your answers.

   (a) $x^2 - 2x + 4 = 0$
   (b) $x^2 - 4x + 5 = 0$

3. Find the value of i^n for every positive integer n, where $i^2 = -1$, $i^3 = i^2 i$, $i^4 = i^2 i^2$ etc.

1.2 Complex Number (x, y)

As we have just seen in Section 1.1, there are numbers like $x_1 + y_1 i$ and $x_2 + y_2 i$ such that

$$\begin{aligned}
(x_1 + y_1 i) + (x_2 + y_2 i) &= (x_1 + x_2) + (y_1 + y_2) i \qquad \text{and} \\
(x_1 + y_1 i)(x_2 + y_2 i) &= x_1 x_2 + (x_1 y_2 + y_1 x_2) i + y_1 y_2 i^2 \\
&= x_1 x_2 + (x_1 y_2 + y_1 x_2) i + y_1 y_2 (-1) \\
&= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i
\end{aligned}$$

for real numbers x_1, x_2, y_1, y_2, so we define the number $x + yi$ as follows.

A **complex number** is an ordered pair (x, y) of real numbers with equality and operations— sum and product— of two complex numbers (x_1, y_1), (x_2, y_2) are defined as follows:

Equality $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$

Sum $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Product $(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$

Then we have

$$\boxed{(x, 0) + (y, 0) = (x + y, 0) \quad \text{and} \quad (x, 0)(y, 0) = (xy, 0)}$$

From these facts, all real numbers can be considered as complex numbers as shown in the following table.

Real Numbers	Complex Numbers
x	$(x, 0)$
y	$(y, 0)$
$x + y$	$(x, 0) + (y, 0)$
xy	$(x, 0)(y, 0)$

Since $(0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0)$, let us denote

$$\boxed{i = (0, 1)}$$

and with the convention $i^2 = ii$, $i^3 = i^2 i$, etc., we have

$$\boxed{i^2 = (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1}$$

And we have

$$\begin{aligned} x + yi &= (x, 0) + (y, 0)(0, 1) \\ &= (x, 0) + (0 - 0, y + 0) \\ &= (x, 0) + (0, y) \\ \end{aligned}$$

$$\boxed{x + yi = (x, y)}$$

Note that sum and product of complex numbers satisfy commutative, associative and distributive properties.

Example 3.

Compute $(-2, 3)(1, -2) + (1, 1)(0, 1)$.

Solution

Method 1

$$\begin{aligned} &(-2, 3)(1, -2) + (1, 1)(0, 1) \\ &= (-2 - (-6), 4 + 3) + (0 - 1, 1 + 0) \\ &= (4, 7) + (-1, 1) \\ &= (3, 8) \end{aligned}$$

Method 2

$$\begin{aligned} &(-2, 3)(1, -2) + (1, 1)(0, 1) \\ &= (-2 + 3i)(1 - 2i) + (1 + i)i \\ &= (-2 + 4i + 3i - 6i^2) + (i + i^2) \\ &= (-2 + 7i - 6(-1)) + (i - 1) \\ &= (4 + 7i) + (-1 + i) \\ &= 3 + 8i \\ &= (3, 8) \end{aligned}$$

Exercise 1.2

1. Compute:
   (a) $(2, 0)(3, 5) + (3, -2)(0, 1)$
   (b) $(2, -5)(-1, 0) + (1, 0)(5, 1)$
   (c) $(-3, -2)(-2, -3) + (-2, -3)(-3, -2)$
   (d) $(1, 0)(0, 1) + (0, 1)(1, 0)$.

2. Compute:
   (a) $(3 + 2i)(3 - 2i) + (-5 + 7i)(-1 - i)$
   (b) $(-1 + i)(1 - i) + (2 + 3i)$
   (c) $(1 + i)(1 - i) + (-2 + i)(-2 + i)$
   (d) $(3 + 2i) + (7 - i)(-3 + 3i)$.

1.3 Operations on Complex Numbers

Sum and product of complex numbers were defined in Section 1.2. Subtraction of complex numbers is defined as in real numbers as follows:

$$\boxed{(x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 + y_1 i) + (-x_2 - y_2 i)}$$

Therefore sum, product and subtraction of complex numbers can be performed as in the real numbers, except only that $i^2 = -1$. But the division of complex numbers is a little different from the division of real numbers. We need some notations to define the division.

From now on let us denote a complex number by z, so that

$$\boxed{z = x + yi = (x, y)}$$

The **conjugate** $\overline{z}$ of a complex number $z = x + yi = (x, y)$ is defined by

$$\boxed{\overline{z} = x - yi = (x, -y)}$$

Then we have

$$\boxed{z\overline{z} = (x + yi)(x - yi) = x^2 + y^2}$$

Let $z_1 = x_1 + y_1 i$ and $z_2 + y_2 i$. We will calculate $\frac{z_1}{z_2}$, $(z_2 \neq 0)$, as follows:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1}{z_2}\frac{\overline{z}_2}{\overline{z}_2} \\
&= \frac{x_1 + y_1 i}{x_2 + y_2 i}\frac{x_2 - y_2 i}{x_2 - y_2 i} \\
&= \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - x_1 y_2)i}{{x_2}^2 + {y_2}^2} \\
&= \frac{x_1 x_2 + y_1 y_2}{{x_2}^2 + {y_2}^2} + \frac{y_1 x_2 - x_1 y_2}{{x_2}^2 + {y_2}^2} i
\end{aligned}$$

Example 4.

Calculate $\frac{2 + 3i}{3 + i}$.

Solution

$$\begin{aligned}
\frac{2 + 3i}{3 + i} &= \frac{2 + 3i}{3 + i}\frac{3 - i}{3 - i} \\
&= \frac{6 + 3 + (9 - 2)i}{9 + 1} \\
&= \frac{9 + 7i}{10} \\
&= \frac{9}{10} + \frac{7}{10} i
\end{aligned}$$

Now we calculate $\frac{1}{z}$ for $z = x + yi$, $z \neq 0$.

$$\begin{aligned}\frac{1}{z} &= \frac{1}{x+yi}\,\frac{x-yi}{x-yi}\\ &= \frac{x-yi}{x^2+y^2}\\ &= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\,i\end{aligned}$$

And we can check that

$$\begin{aligned}z\,(\frac{1}{z}) &= (x+yi)(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\,i)\\ &= \frac{x^2+y^2}{x^2+y^2} + \frac{yx-xy}{x^2+y^2}\,i\\ &= 1\end{aligned}$$

So, for a non-zero complex number z, $\frac{1}{z}$ is the **multiplicative inverse** of z and is denoted by z^{-1}. From these facts, division of complex numbers is defined as

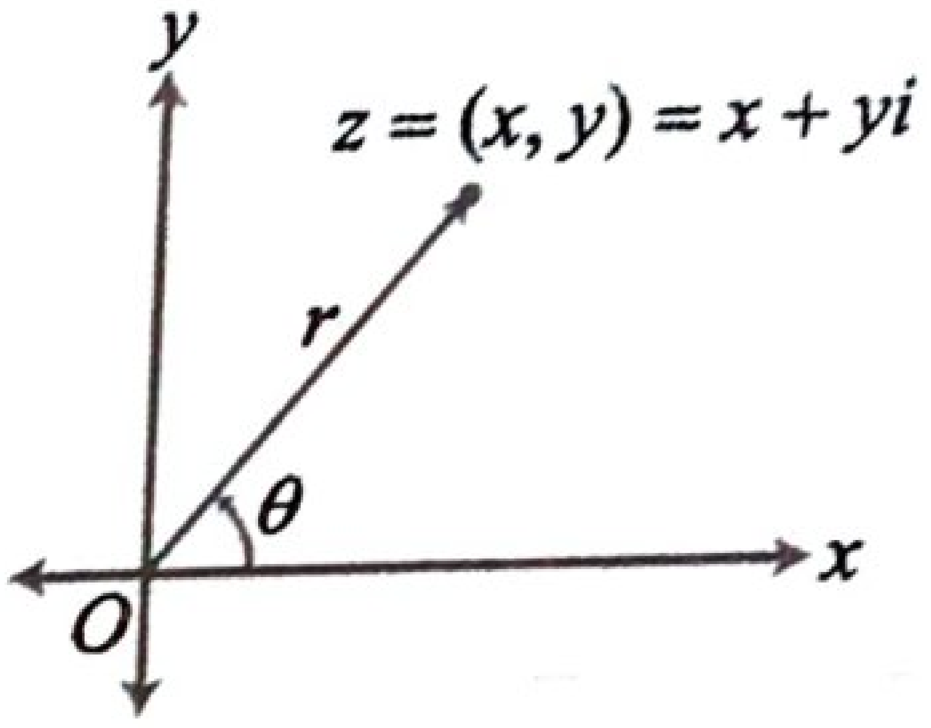
$$\boxed{\frac{z_1}{z_2} = z_1 {z_2}^{-1}}$$

but we usually calculate $\frac{z_1}{z_2}$ as in Example 4.

Exercise 1.3

1. Let $z_1 = -2+3i$, $z_2 = 5+2i$. Compute:
   - (a) $z_1^2 - 2z_1 + 1$
   - (b) $3z_2^2 + 2z_2 - 1$
   - (c) $z_1\overline{z_2} + z_2\overline{z_1}$
   - (d) $\frac{1}{z_1}$
   - (e) $\frac{1}{z_2}$
   - (f) $\frac{1}{z_1 z_2}$
   - (g) $\frac{z_1}{z_2}$
   - (h) $\frac{\overline{z_1}}{\overline{z_2}}$
   - (i) $\frac{z_2}{z_1}$
   - (j) $\overline{\left(\frac{z_2}{z_1}\right)}$
   - (k) $\frac{\overline{z_1}z_2}{z_1\overline{z_2}}$
   - (l) $\frac{z_2}{\overline{z_1}} + \frac{z_1}{\overline{z_2}}$
2. Let $z_1 = 3-2i$, $z_2 = -1+4i$. Show that
   - (a) $\overline{(z_1+z_2)} = \overline{z_1} + \overline{z_2}$
   - (b) $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$
   - (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

1.4 Trigonometric Form



As a complex number $z = x + yi$ is an ordered pair (x, y) of real numbers, we can place z in xy-coordinate plane as usual. If the length of the line segment from O to z is r, we say that the **absolute value** of z is r and is denoted by $|z|$. Here angle θ, measured in radians, is an angle with positive x-axis and the line segment. From trigonometry, we have known that

$$\boxed{r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

So we have $z = (x, y) = x + yi = r\cos\theta + i\,r\sin\theta$,

$$\boxed{z = r(\cos\theta + i\,\sin\theta)}$$

which is called the **trigonometric form** of a complex number. $z = (r, \theta)$ is called the **polar form** of z.

Example 5.

Find the trigonometric form with $-\pi < \theta \leq \pi$.

(a) $z = 1 + \sqrt{3}\,i$ (b) $z = -1 + i$
(c) $z = -\sqrt{3} - i$ (d) $z = -1$

Solution

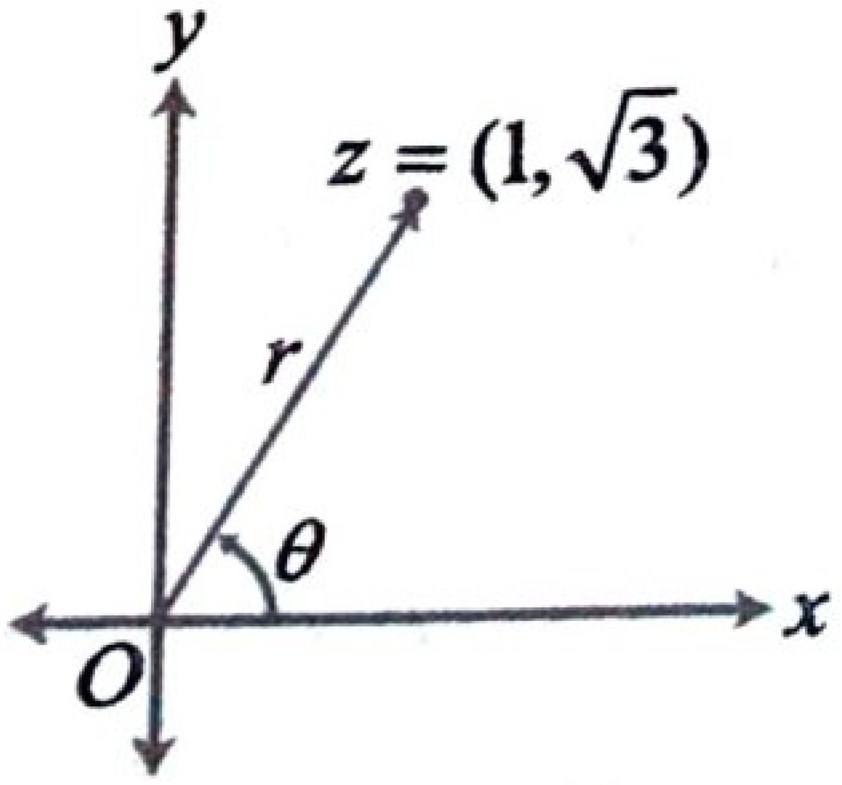
(a)

$$z = 1 + \sqrt{3}\,i = (1, \sqrt{3})$$

$$r = \sqrt{1+3} = 2, \quad \cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

$$z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$



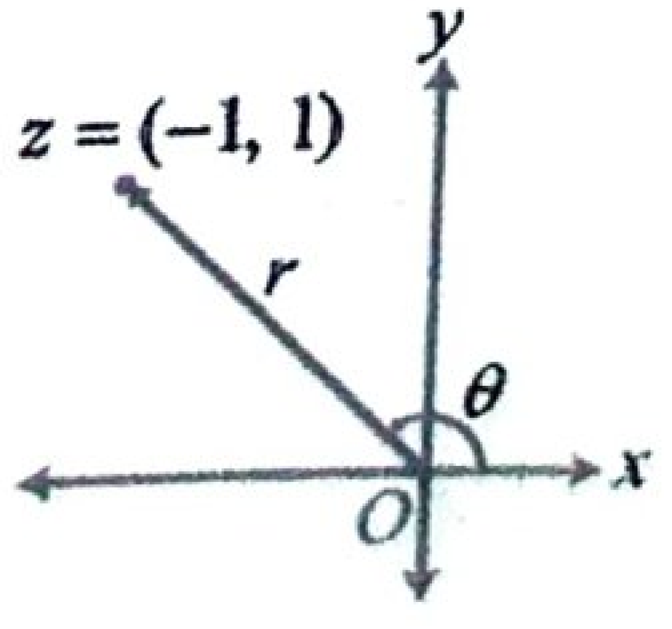
(b)

$$z = -1 + i = (-1, 1)$$

$$r = \sqrt{1+1} = \sqrt{2}, \quad \cos\theta = -\frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{3\pi}{4}$$

$$z = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

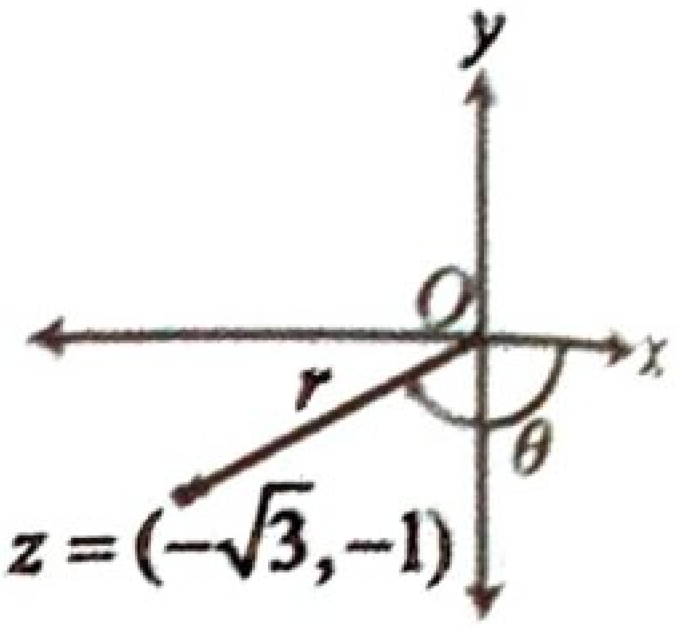


(c) $z = -\sqrt{3} - i = (-\sqrt{3}, -1)$

$r = \sqrt{3+1} = 2, \quad \cos\theta = -\frac{\sqrt{3}}{2}, \quad \sin\theta = -\frac{1}{2}$

$\theta = -\frac{5\pi}{6}$

$z = 2(\cos(-\frac{5\pi}{6}) + i\sin(-\frac{5\pi}{6}))$

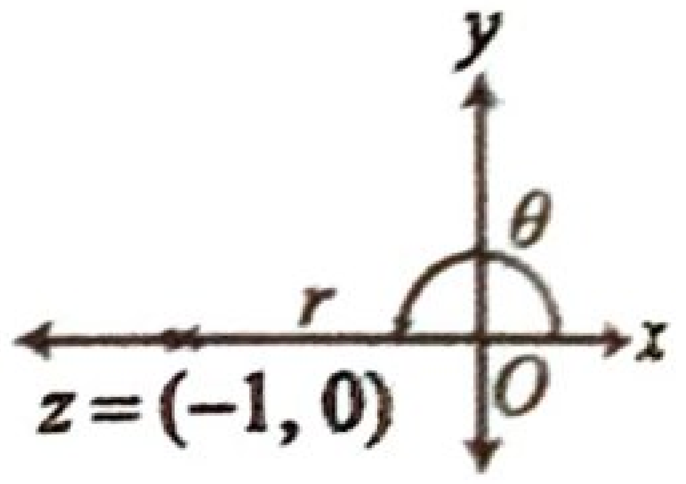


(d) $z = -1 = (-1, 0)$

$r = 1, \quad \cos\theta = -1, \quad \sin\theta = 0$

$\theta = \pi$

$z = \cos\pi + i\sin\pi$



Product in Trigonometric Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then their product is

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\ \sin(\theta_1+\theta_2)) \end{aligned}$$

$$\boxed{z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\ \sin(\theta_1+\theta_2))}$$

Example 6.

Given that $z_1 = 1 + \sqrt{3}i$, $z_2 = -1 + i$, find z_1z_2 by using trigonometric forms. Check your answer by direct multiplication.

Solution

$z_1 = 1 + \sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ (See Example 5(a))

$z_2 = -1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ (See Example 5(b))

$$\begin{aligned} z_1z_2 &= 2\sqrt{2}(\cos(\frac{\pi}{3} + \frac{3\pi}{4}) + i\sin(\frac{\pi}{3} + \frac{3\pi}{4})) \\ &= -1 - \sqrt{3} + (1 - \sqrt{3})\,i \end{aligned}$$

$$\begin{aligned} z_1z_2 &= (1 + \sqrt{3}i)(-1 + i) \\ &= -1 + i - \sqrt{3}i + \sqrt{3}i^2 \\ &= -1 - \sqrt{3} + (1 - \sqrt{3})\,i \end{aligned}$$

Multiplicative Inverse in Trigonometric Form

Since $z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ for $z = x + yi$,

$$\begin{aligned} z^{-1} &= \frac{1}{x^2+y^2}(x - yi) \\ &= \frac{1}{r^2}\,r\,(\cos\theta - i\sin\theta) \\ &= \frac{1}{r}(\cos\theta - i\sin\theta) \end{aligned}$$

$$\boxed{z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))}$$

Example 7.

Given that $z = -\sqrt{3} - i$, using trigonometric form of z, find z^{-1}. Check your answer by showing that $zz^{-1} = 1$.

Solution

$$\begin{aligned} z &= -\sqrt{3} - i \\ z &= 2(\cos(-\frac{5\pi}{6}) + i\sin(-\frac{5\pi}{6})) \qquad \text{(See Example 5(c))} \end{aligned}$$

$$\begin{aligned} z^{-1} &= \frac{1}{2}(\cos\frac{5\pi}{6} + i\sin(\frac{5\pi}{6})) \\ &= \frac{1}{2}(-\frac{\sqrt{3}}{2} + \frac{1}{2}i) \\ &= -\frac{\sqrt{3}}{4} + \frac{1}{4}i \end{aligned}$$

$$zz^{-1} = (-\sqrt{3} - i)(-\frac{\sqrt{3}}{4} + \frac{1}{4}i) = \frac{3}{4} - \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i - \frac{1}{4}i^2 = \frac{3}{4} + \frac{1}{4} = 1$$

Division in Trigonometric Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= r_1(\cos\theta_1 + i\sin\theta_1)\,\frac{1}{r_2}(\cos(-\theta_2) + i\sin(-\theta_2)) \end{aligned}$$

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))}$$

Example 8.

Given that $z_1 = 1 + \sqrt{3}\,i$, $z_2 = -1 + i$, find $\frac{z_1}{z_2}$ by using trigonometric forms. Check your answer by direct calculation.

Solution

$$z_1 = 1 + \sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \qquad \text{(See Example 5(a))}$$

$$z_2 = -1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) \qquad \text{(See Example 5(b))}$$

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2}{\sqrt{2}}(\cos(\frac{\pi}{3} - \frac{3\pi}{4}) + i\sin(\frac{\pi}{3} - \frac{3\pi}{4})) \\
&= \frac{-1+\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i \\
\frac{z_1}{z_2} &= \frac{1+\sqrt{3}\,i}{-1+i}\,\frac{-1-i}{-1-i} \\
&= \frac{-1-i-\sqrt{3}\,i-\sqrt{3}\,i^2}{1+1} \\
&= \frac{-1+\sqrt{3}-(1+\sqrt{3})i}{2} \\
&= \frac{-1+\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i
\end{aligned}$$

Powers of Complex Numbers

The powers of a complex number $z = r(\cos\theta + i\sin\theta)$ is given by

$$\boxed{z^n = r^n(\cos n\theta + i\,\sin n\theta), \quad n \text{ is an integer.}}$$

Example 9.

Given that $z = 1 + \sqrt{3}\,i$, find (a) z^{10} (b) z^{-10}.

Solution

$$z = 1 + \sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \qquad \text{(See Example 5(a))}$$

(a) $z^{10} = 2^{10}(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}) = 1024(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = -512 - 512\sqrt{3}\,i$

(b) $z^{-10} = \frac{1}{2^{10}}(\cos(-\frac{10\pi}{3}) + i\sin(-\frac{10\pi}{3})) = \frac{1}{1024}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = -\frac{1}{2048} + \frac{\sqrt{3}}{2048}i$

Exercise 1.4

1. Find the trigonometric form with $-\pi < \theta \leq \pi$.

   (a) $z = 1 - \sqrt{3}\,i$ (b) $z = -\sqrt{2} + \sqrt{2}\,i$ (c) $z = -2 - 2i$

   (d) $z = \sqrt{3} - i$ (e) $z = i$ (f) $z = -3i$

2. Given that $z_1 = 2 - 2\sqrt{3}\,i$, $z_2 = -1 - i$, find the following complex numbers by using trigonometric forms. Check your answer by direct calculation.

   (a) $z_1 z_2$ (b) z_1^{-1} (c)z_2^{-1} (d) $\frac{z_1}{z_2}$ (e) $\frac{z_2}{z_1}$

3. Given that $z = -2\sqrt{3} - 2i$, find (a) z^5 (b) z^{-5}.

1.5 Roots of Complex Numbers

An **n^{th} root of a complex number z** is a complex number w that satisfies the equation

$$\boxed{w^n = z}$$

Since $z = r(\cos\theta + i\sin\theta)$ and

$$(\sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}))^n = (\sqrt[n]{r})^n(\cos n(\frac{\theta}{n}) + i\sin n(\frac{\theta}{n})) = r(\cos\theta + i\sin\theta) = z$$

we have an n^{th} root of a complex number z as $\sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$.

Moreover, since z can also be represented as

$$z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$$

for every integer k, we have

$$\begin{aligned}(\sqrt[n]{r}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}))^n &= (\sqrt[n]{r})^n(\cos n\frac{\theta + 2k\pi}{n} + i\sin n\frac{\theta + 2k\pi}{n}) \\ &= r(\cos(\theta + 2k\pi) + i(\sin\theta + 2k\pi)) \\ &= z\end{aligned}$$

So n^{th} roots of z are

$$\sqrt[n]{r}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n})$$

for every integer k.

But for $k = n$, we have

$$\begin{aligned}\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n} &= \cos\frac{\theta+2n\pi}{n}+i\sin\frac{\theta+2n\pi}{n}\\ &= \cos(\frac{\theta}{n}+2\pi)+i\sin(\frac{\theta}{n}+2\pi)\\ &= \cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\end{aligned}$$

so the root for $k = n$ is the same as the root for $k = 0$. The same is also true for $k >$ as the root for $k = n + 1$ is same as the root for $k = 1$, and so on. Therefore the ro of z, denoted by w_k of z are

$$\boxed{w_k = \sqrt[n]{r}(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}), \qquad k = 0, 1, 2, \ldots, n-1.}$$

Example 10.

Find the cube roots of $z = -2 - 2i$.

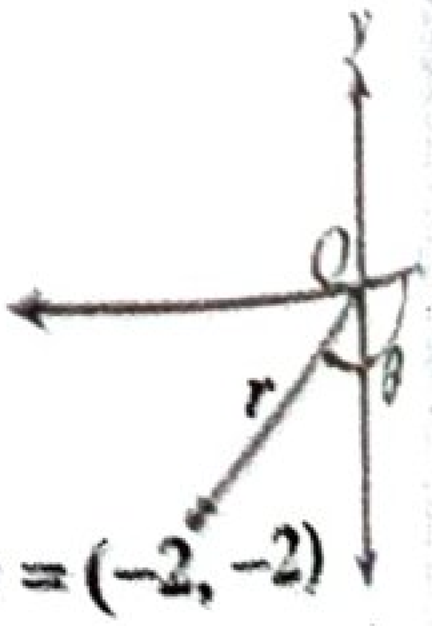
Solution

$$z = -2 - 2i = (-2, -2)$$

$$r = \sqrt{4+4} = 2\sqrt{2}, \quad \cos\theta = -\frac{1}{\sqrt{2}}, \quad \sin\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = -\frac{3\pi}{4}$$

$$z = 2\sqrt{2}(\cos(-\frac{3\pi}{4})+i\sin(-\frac{3\pi}{4}))$$



Then we have

$$w_k = \sqrt[3]{2\sqrt{2}}\,(\cos\frac{-\frac{3\pi}{4}+2k\pi}{3}+i\sin\frac{-\frac{3\pi}{4}+2k\pi}{3}), \qquad k = 0, 1, 2.$$

Therefore the cube roots are

$$w_0 = \sqrt{2}\,(\cos(-\frac{3\pi}{12})+i\sin(-\frac{3\pi}{12})) = \sqrt{2}\,(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}) = 1-i$$

$$w_1 = \sqrt{2}\,(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}) = \sqrt{2}\,(\frac{\sqrt{6}-\sqrt{2}}{4}+i\frac{\sqrt{6}+\sqrt{2}}{4}) = \frac{-1+\sqrt{3}}{2}+\frac{1+}{2}$$

$$\begin{aligned}w_2 &= \sqrt{2}\,(\cos(\frac{13\pi}{12})+i\sin(\frac{13\pi}{12})) = \sqrt{2}\,(-\frac{\sqrt{6}+\sqrt{2}}{4}+i\frac{-\sqrt{6}+\sqrt{2}}{4})\\ &= -\frac{1+\sqrt{3}}{2}+\frac{1-\sqrt{3}}{2}i\end{aligned}$$

Example 11.

Solve $z^6 = 1$.

Solution

$$\begin{aligned}
1 &= (1,0) = 1(\cos 0 + i\,\sin 0) = 1(\cos(0+2k\pi) + i\,\sin(0+2k\pi))\\
z^6 &= 1\\
z^6 &= 1(\cos(0+2k\pi) + i\sin(0+2k\pi))\\
z &= (1(\cos(0+2k\pi) + i\sin(0+2k\pi))^{\frac{1}{6}}\\
z &= \sqrt[6]{1}(\cos\frac{0+2k\pi}{6} + i\sin\frac{0+2k\pi}{6}), \qquad k = 0, 1, 2, \ldots, 5
\end{aligned}$$

When $k = 0, z = \cos 0 + i\,\sin 0 = 1.$

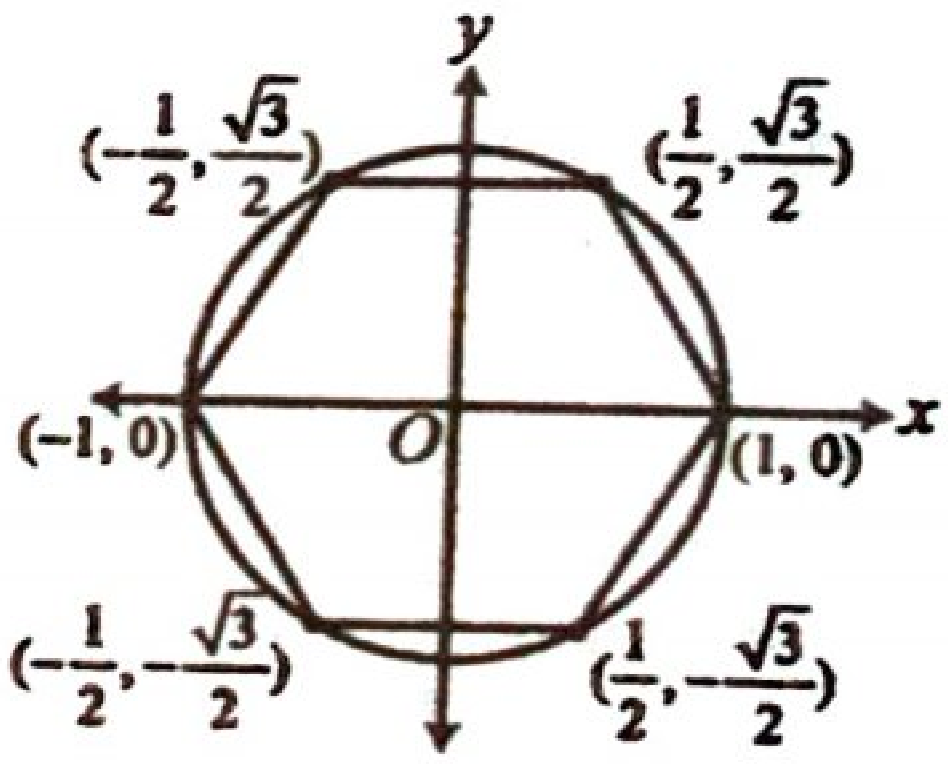
When $k = 1, z = \cos\frac{\pi}{3} + i\,\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$

When $k = 2, z = \cos\frac{2\pi}{3} + i\,\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$

When $k = 3, z = \cos\pi + i\,\sin\pi = -1.$

When $k = 4, z = \cos\frac{4\pi}{3} + i\,\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$

When $k = 5, z = \cos\frac{5\pi}{3} + i\,\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$


y
$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
(−1, 0)
O
(1, 0)
x
$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Exercise 1.5

1. Find the square roots of the following complex numbers.

   (a) $1 + \sqrt{3}i$ (b) i (c) $-\sqrt{3} + i$

   (d) $-1 - \sqrt{3}i$ (e) $-i$ (f) $\sqrt{3} - i$

2. Find the cube roots of the following complex numbers.

   (a) $1 + i$ (b) i (c) $-1 + i$

   (d) $-1 - i$ (e) $-i$ (f) $1 - i$

3. Solve the following equations.

   (a) $z^4 = -i$ (b) $z^4 = -1$ (c) $z^4 = -8 - 8\sqrt{3}i$ (d) $z^6 = -1$

Chapter 2

Mathematical Induction

2.1 Introduction

Mathematical induction is a powerful and elegant technique for proving certa
f mathematical statements, general propositions which assert that something is
or all natural numbers or for all natural numbers from some point on. One of
echniques to prove mathematical statements discussed in this chapter is the Prin
f Mathematical Induction. The actual term mathematical induction was first use
De Morgan, even though the method was used by Fermat, Pascal and others be
im. The method is used in many branches of Higher Mathematics.

Before we state the principle of mathematical induction, let us consider an exa
Consider the sum of the first n odd positive integers. That is,

$$\begin{array}{llllll}
\text{if} & n=1, & 1=1^2, & & & \\
\text{if} & n=2, & 1+3 & & & \\
\text{if} & n=3, & 1+3+5 & = & 4 & = 2^2, \\
\text{if} & n=4, & 1+3+5+7 & = & 9 & = 3^2, \\
\text{if} & n=5, & 1+3+5+7+9 & = & 16 & = 4^2, \\
\text{if} & n=6, & 1+3+5+7+9+11 & = & 25 & = 5^2, \\
 & & & = & 36 & = 6^2.
\end{array}$$

From the results above, it looks as if the sum of the first n odd natural numbe
lways given by n^2. To express this statement symbolically, first observe that th
dd natural number is $2n-1$. Then the statement can be expressed as:

$$1+3+5+7+9+\cdots+(2n-1)=n^2.$$

Although from this pattern we might conjecture that statement (1) is tr
ny choice of n, can we really be sure that it does not fail for some choice of
ave seen that the statement is true for $n = 1, 2, 3, 4, 5, 6$ by direct calculati
he statement true for any natural number n? Actually, no matter how man

we check, we can never prove that the statement is always true because there are infinitely many cases and direct calculation cannot check them all. The method of proof by mathematical induction will, in fact, prove that any mathematical statement like (1) is true for all natural numbers n.

2.2 Principle of Mathematical Induction

For each natural number n, let $P(n)$ be a statement depending on n. Suppose that the following two conditions are satisfied.

1. The statement is true for $n = 1$.
2. For any natural number k, if the statement is true for $n = k$, then the statement is true for $n = k + 1$.

Then the statement $P(n)$ is true for all natural numbers n.

To apply this principle, there are two steps:

The initial step: Prove that the statement is true for $n = 1$.

The inductive step: Assume that the statement is true for $n = k$, and use this assumption to prove that the statement is true for $n = k + 1$.

Conclusion: The statement has been proved for $n = 1$. By the inductive step, since the statement is true for $n = 1$, it is also true for $n = 2$. Again, by the inductive step, since the statement is true for $n = 2$, it is also true for $n = 3$. And since the statement is true for $n = 3$, it is also true for $n = 4$, and so on. Finally, we conclude that the statement is true for all natural numbers n.

Remark. In the inductive step we do not prove that the statement is true for $n = k$. We only show that if the statement is true for $n = k$, then the statement is also true for $n = k + 1$. The assumption that the statement is true for $n = k$, is called ***the induction hypothesis***.

Example 1.

Use the mathematical induction principle to prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all natural numbers n.

Solution

Let $P(n)$ denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(1) For $n = 1$, L.H.S. $= 1$,
R.H.S. $= 1^2 = 1$.

L.H.S. = R.H.S.

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

We will show that the statement is true for $n = k+1$, that is

we have to show that $1+3+5+\ldots+(2k-1)+[2(k+1)-1] = (k+1)^2$.

It is proved that
$$\begin{aligned}1+3+5+\cdots+(2k-1)+[2(k+1)-1] &= k^2+[2(k+1)-1]\\ &= k^2+2k+2-1\\ &= k^2+2k+1\\ &= (k+1)^2.\end{aligned}$$

Therefore, the statement is true for $n = k+1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Example 2.

Use the mathematical induction principle to prove that $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$, for all natural numbers n.

Solution

Let $P(n)$ denote the statement $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$.

(1) For $n = 1$, L.H.S. $= 1$,

R.H.S. $= \dfrac{1(1+1)}{2} = 1$.

L.H.S. = R.H.S.

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

We will show that the statement is true for $n = k+1$, that is

we have to show that $1+2+3+\cdots+k+(k+1) = \dfrac{(k+1)((k+2)}{2}$.

It is proved that
$$\begin{aligned}1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\ &= \frac{k^2+k+2k+2}{2}\end{aligned}$$

$$1+2+3+\cdots+k+(k+1)=\frac{k^2+3k+2}{2}$$
$$=\frac{(k+1)(k+2)}{2}.$$

Therefore, the statement is true for $n=k+1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

In some cases, a statement involving a variable n holds when the natural numbers $n \geq m$, $m \in \mathbb{N}$ and the statement does not hold when $n < m$. In this case, we will prove that the statement is true for $n = m$ in the initial step.

Example 3.

Use the mathematical induction principle to prove that $3^n - 1$ is a multiple of 2 for all natural numbers n.

Solution

Let $P(n)$ denote the statement $3^n - 1$ is a multiple of 2.

(1) For $n = 1$, $3^1 - 1 = 2$ which is a multiple of 2.

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$3^k - 1$ is a multiple of 2.

We will show that the statement is true for $n = k+1$, that is

we have to show that $3^{k+1} - 1$ is a multiple of 2.

It is proved that $3^{k+1} - 1 = 3 \cdot 3^k - 1 = 2 \cdot 3^k + 3^k - 1$.

Since $2 \cdot 3^k$ is a multiple of 2 and $3^k - 1$ is a multiple of 2, we can say that $2 \cdot 3^k + 3^k - 1$ is also a multiple of 2. So, $3^{k+1} - 1$ is also a multiple of 2.

Therefore, the statement is true for $n = k+1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Example 4.

Prove that $(ab)^n = a^n b^n$ for every natural number n.

Solution

Let $P(n)$ denote the statement $(ab)^n = a^n b^n$.

(1) For $n = 1$. L.H.S. $= (ab)^1 = ab$.

$$\text{R.H.S.} = a^1b^1 = ab.$$

$$\text{L.H.S.} = \text{R.H.S.}$$

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$$(ab)^k = a^kb^k.$$

We will show that the statement is true for $n = k + 1$, that is

we have to show that $(ab)^{k+1} = a^{k+1}b^{k+1}$.

It is proved that $(ab)^{k+1} = (ab)^k(ab) = (a^kb^k)(ab) = (a^ka)(b^kb) = a^{k+1}b^{k+1}$.

Therefore, the statement is true for $n = k + 1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for every natural number n.

Example 5.

Prove that $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{(2n-1)3^{n+1} + 3}{4}$ for all natural numbers n by the use of the mathematical induction principle .

Solution

Let $P(n)$ denote the statement $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \ldots + n \cdot 3^n = \frac{(2n-1)3^{n+1} + 3}{4}$.

(1) For $n = 1$, L.H.S. $= 1 \cdot 3 = 3$,

$$\text{R.H.S.} = \frac{(2-1)3^2 + 3}{4} = \frac{12}{4} = 3.$$

$$\text{L.H.S.} = \text{R.H.S.}$$

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$$1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k = \frac{(2k-1)3^{k+1} + 3}{4}.$$

We will show that the statement is true for $n = k + 1$, that is

we have to show that

$$1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k + (k+1) \cdot 3^{k+1} = \frac{(2(k+1) - 1)3^{k+2} + 3}{4}.$$

It is proved that

$$1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k + (k+1) \cdot 3^{k+1} = \frac{(2k-1)3^{k+1} + 3}{4} + (k+1) \cdot 3^{k+1}$$

$$\begin{aligned}1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+k\cdot 3^k+(k+1)\cdot 3^{k+1} &= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}\\ &= \frac{(6k+3)3^{k+1}+3}{4}\\ &= \frac{(2k+1)3^{k+2}+3}{4}\\ &= \frac{(2(k+1)-1)3^{k+2}+3}{4}.\end{aligned}$$

Therefore, the statement is true for $n = k + 1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Example 6.

Use the mathematical induction principle to prove that $a - b$ is a factor of $a^n - b^n$ for all natural numbers n.

Solution

Let $P(n)$ denote the statement $a - b$ is a factor of $a^n - b^n$.

(1) For $n = 1$, $a^1 - b^1 = a - b$.

So, $a - b$ is a factor of $a^1 - b^1$.

The statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is

$a - b$ is a factor of $a^k - b^k$.

We will show that the statement is true for $n = k + 1$, that is

we have to show that $a - b$ is a factor of $a^{k+1} - b^{k+1}$.

It is proved that

$$\begin{aligned}a^{k+1} - b^{k+1} &= a^{k+1} - a^k b + a^k b - b^{k+1}\\ &= a^k(a-b) + b(a^k - b^k).\end{aligned}$$

Since $a - b$ is a factor of $a^k(a-b)$ and $a - b$ is a factor of $a^k - b^k$, $a - b$ is also a factor of $a^k(a-b) + b(a^k - b^k)$. So, $a - b$ is a factor of $a^{k+1} - b^{k+1}$.

Therefore, the statement is true for $n = k + 1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Example 7.

Use the mathematical induction principle to prove that $4n < 2^n$ for all natural numbers $n \geq 5$.

Solution

Let $P(n)$ denote the statement $4n < 2^n$.

(1) For $n = 5$, L.H.S. $= 4(5) = 20$,

R.H.S. $= 2^5 = 32$.

L.H.S. $<$ R.H.S.

The statement is true for $n = 5$.

(2) Assume that the statement is true for $n = k \geq 5$, that is

$$4k < 2^k.$$

We will show that the statement is true for $n = k + 1$, that is we have to show that $4(k + 1) < 2^{k+1}$.

It is proved that
$$\begin{aligned} 4(k+1) &= 4k + 4 \\ &< 2^k + 4 \\ &< 2^k + 4k \quad (\text{since } 4 < 4k) \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k = 2^{k+1}. \end{aligned}$$

Hence $4(k+1) < 2^{k+1}$.

Therefore, the statement is true for $n = k + 1$.

(3) Hence, by principle of mathematical induction, the statement $P(n)$ is true for all natural numbers $n \geq 5$.

Exercise 2.1

1. Prove the following by using the principle of mathematical induction for all natural numbers n:

   (a) $1 + 3 + 3^2 + \cdots + 3^{n-1} = \dfrac{3^n - 1}{2}$

   (b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (\dfrac{n(n+1)}{2})^2$

   (c) $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$

   (d) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

   (e) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

   (f) $1 + 2 + 2^2 + + \cdots + 2^{n-1} = 2^n - 1$

   (g) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

2. Prove that 3 is a factor of $4^n - 1$ for all natural numbers n by using the mathematical induction.

3. Prove that $n^3 - n + 3$ is divisible by 3 for all natural numbers n by using the mathematical induction.

4. Prove that $3^{2n} - 1$ is divisible by 8 for all natural numbers n by using the mathematical induction.

5. Prove that $(n+1)^2 < 2n^2$ for all natural numbers $n \geq 3$ by using the mathematical induction.

6. Prove that $(2n+7) < (n+3)^2$ for all natural numbers n by using the mathematical induction.

7. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all natural numbers n by using the mathematical induction.

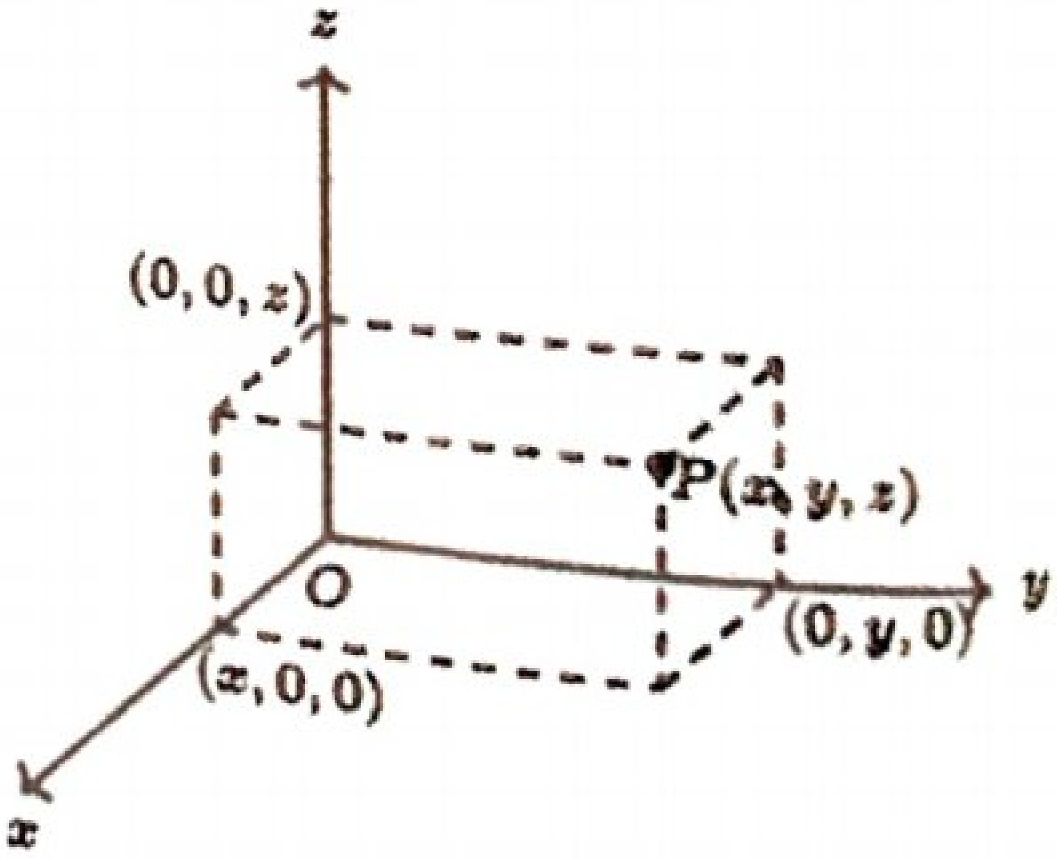
Chapter 3

Analytical Solid Geometry

We have learned about points and lines in two-dimensional rectangular coordinate system. In this chapter, we will extend the system to three dimensions.

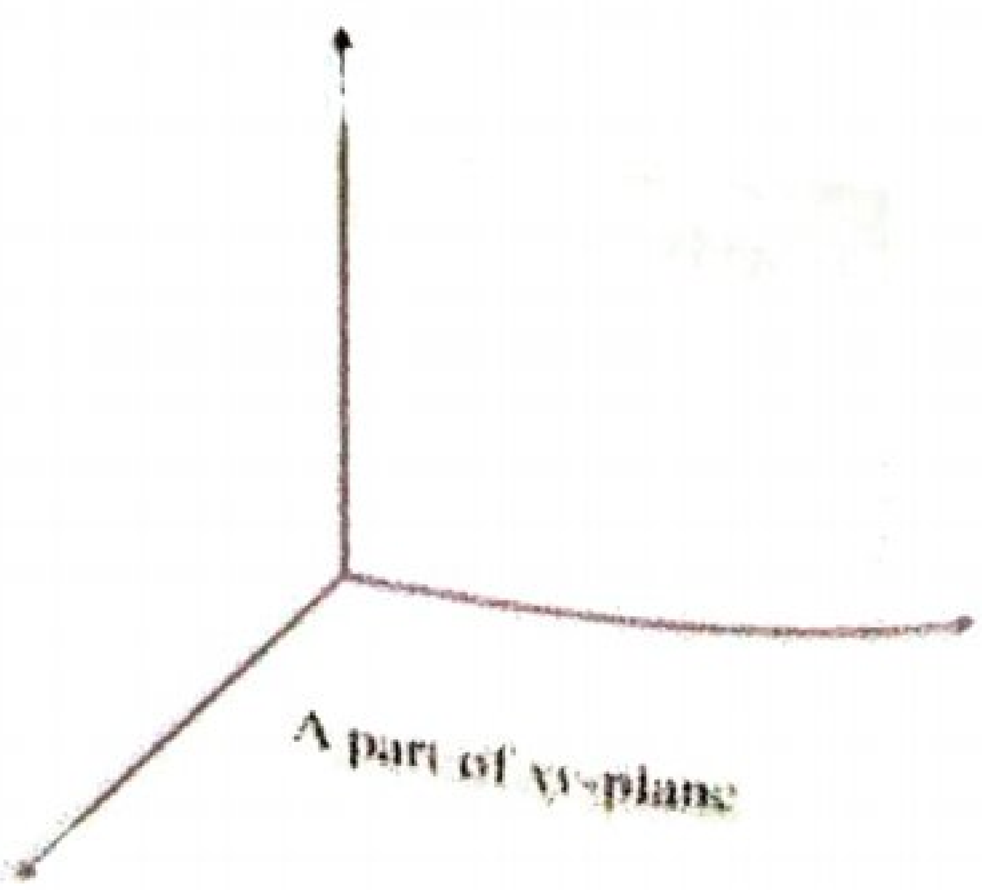
3.1 Coordinates of a Point in Space

In the plane, each point is associated with an ordered pair of real numbers. In space, each point is associated with an ordered triple of real numbers. Through a fixed point, called the **origin** O, draw three mutually perpendicular lines: the x-axis, the y-axis and the z-axis. A point P in space is determined by an ordered triple (x, y, z) of real numbers as shown in the diagram. These numbers x, y, z are called the **coordinates** of P.



The xy**-plane** consists of the x-axis and the y-axis, and the z-axis is perpendicular to the xy-plane. The coordinates of the points on the xy-plane are of the form $(x, y, 0)$.

The yz**-plane** consists of the y-axis and the z-axis, and the x-axis is perpendicular to the yz-plane. The coordinates of the points on the yz-plane are of the form $(0, y, z)$.



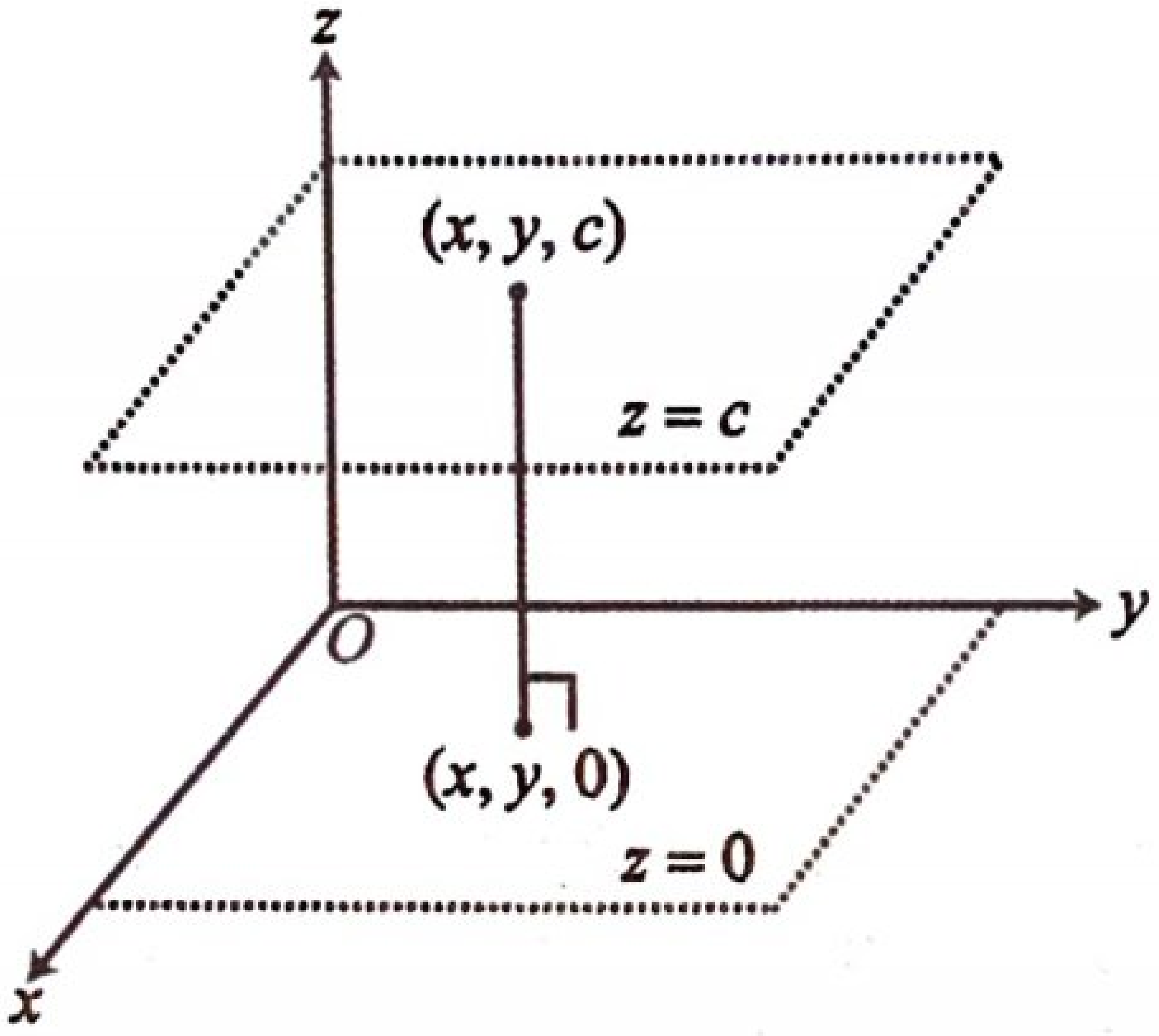
A part of yz-plane

The zx**-plane** consists of the x-axis and the z-axis, and the y-axis is perpendicular to the zx-plane. The coordinates of the points on the zx-plane are of the form $(x, 0, z)$.

The equations of xy-plane, yz-plane and zx-plane are as shown in the following table.

Plane	Equation	Coordinates
xy-plane	$z=0$	$(x,y,0)$
yz-plane	$x=0$	$(0,y,z)$
zx-plane	$y=0$	$(x,0,z)$

Planes parallel to xy-plane: Equation of any plane parallel to the xy-plane is $z=c$, and the coordinates of the points on that plane are of the form(x,y,c).

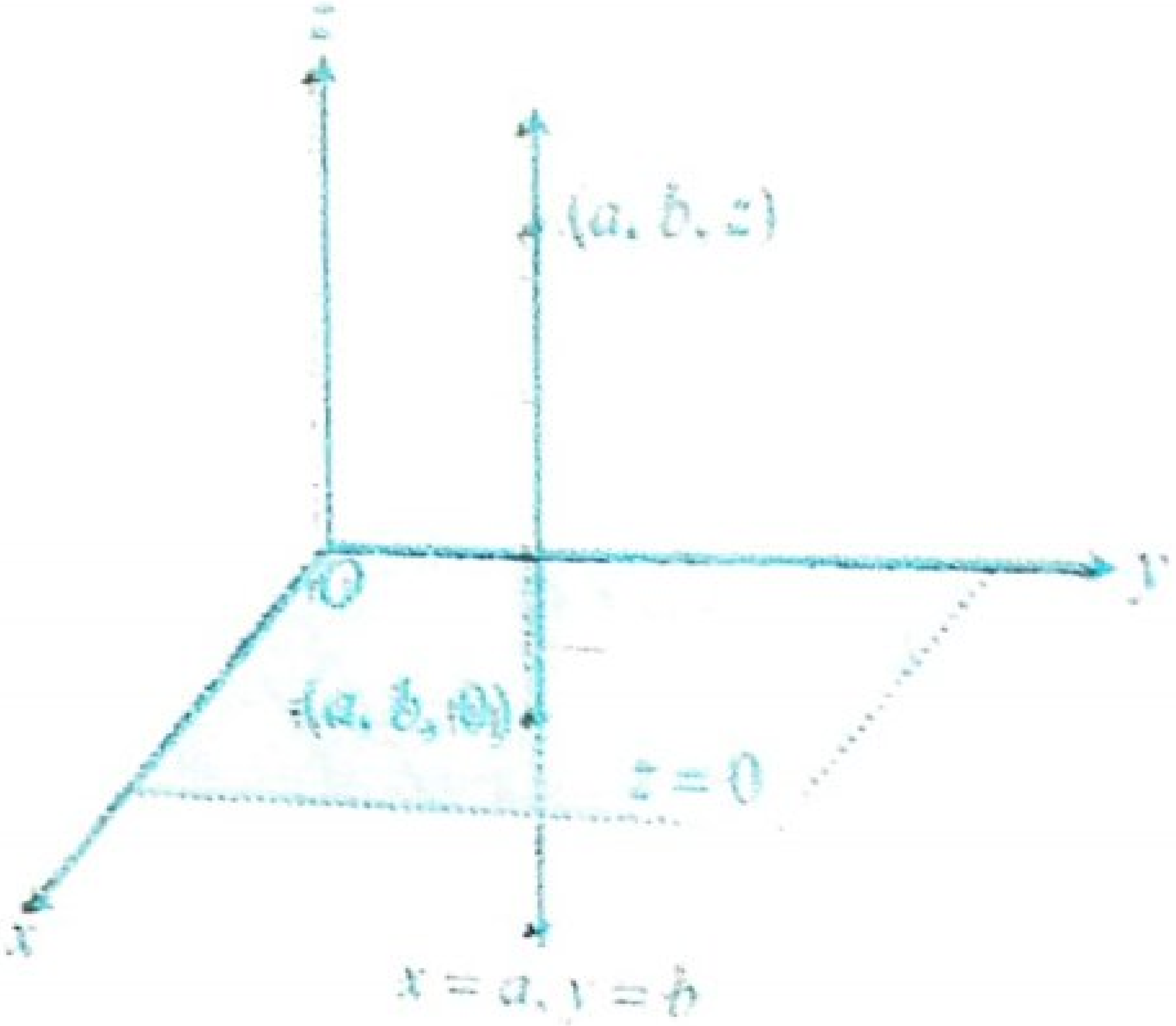


Planes parallel to yz-plane: Equation of any plane parallel to the yz-plane is $x=a$, and the coordinates of the points on that plane are of the form (a,y,z).

Planes parallel to zx-plane: Equation of any plane parallel to the zx-plane is $y=b$, and the coordinates of the points on that plane are of the form (x,b,z).

Plane	Equation	Coordinates	Example	
			Equation	Coordinates
Parallel to xy-plane	$z=c$	(x,y,c)	$z=1$	$(x,y,1)$
Parallel to yz-plane	$x=a$	(a,y,z)	$x=5$	$(5,y,z)$
Parallel to zx-plane	$y=b$	(x,b,z)	$y=2$	$(x,2,z)$

Lines perpendicular to xy-plane: Equation of any line perpendicular to xy-plane and passing through the point (a, b, c) is $x = a, y = b$, and the coordinates the points on that line are of the form (a, b, z).



Lines perpendicular to yz-plane: Equation of any line perpendicular to yz-plane and passing through the point (a, b, c) is $y = b, z = c$, and the coordinates the points on that line are of the form (x, b, c).

Lines perpendicular to zx-plane: Equation of any line perpendicular to zx-plane and passing through the point (a, b, c) is $x = a, z = c$, and the coordinates the points on that line are of the form (a, y, c).

Example 1.

Find the equation of the line through the point $(-3, 5, 7)$ and perpendicular to

(a) xy-plane (b) yz-plane (c) zx-plane.

Find the point of intersection of the line and plane.

Solution

(a) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to xy-pla is

$$x = -3, y = 5 \quad \text{or} \quad (-3, 5, z).$$

The point of intersection of the line and xy-plane is $(-3, 5, 0)$.

(b) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to yz-pla is

$$y = 5, z = 7 \quad \text{or} \quad (x, 5, 7).$$

The point of intersection of the line and yz-plane is $(0, 5, 7)$.

(c) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to zx-plane is

$$x = -3, z = 7 \quad \text{or} \quad (-3, y, 7).$$

The point of intersection of the line and zx-plane is $(-3, 0, 7)$.

Note that

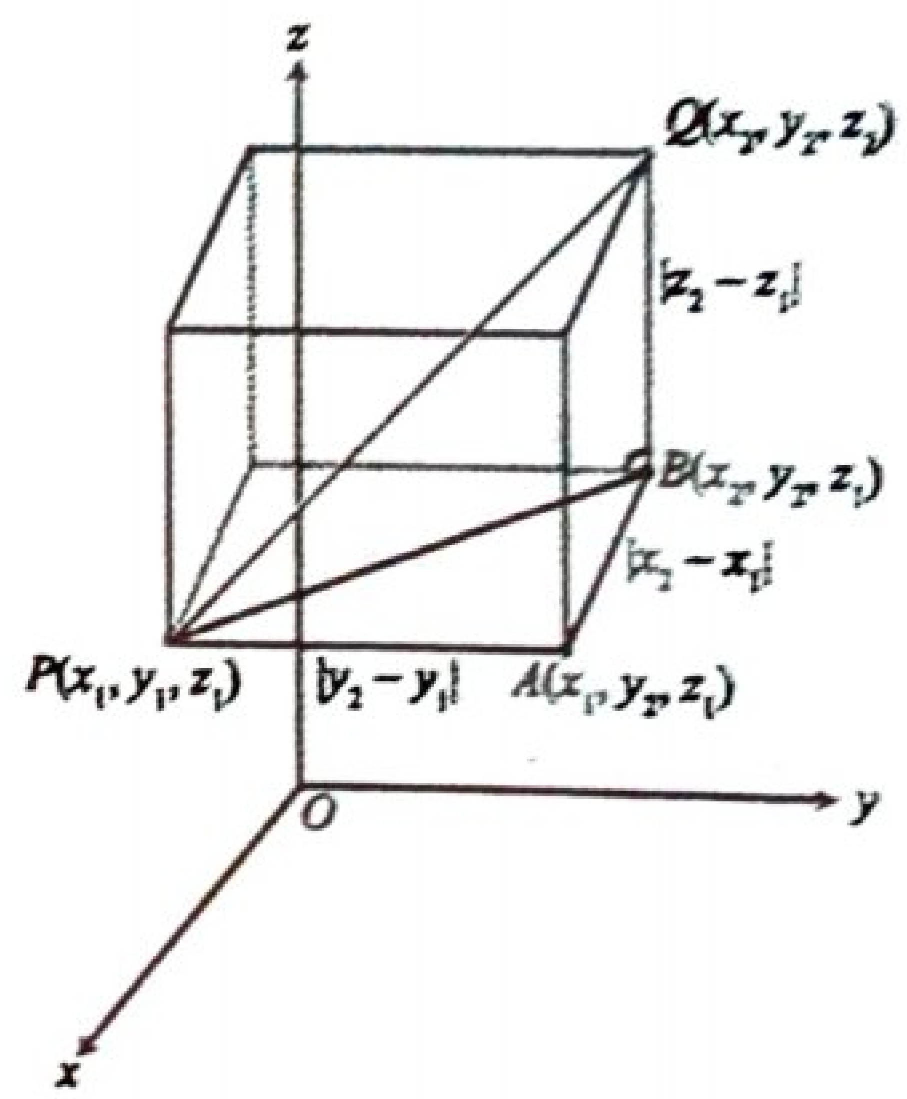
- equation of x-axis: $y = 0, z = 0$
- equation of y-axis: $x = 0, z = 0$
- equation of z-axis: $x = 0, y = 0$

Distance between two points: From the right triangle PAB, we have $PB^2 = AB^2 + PA^2$. Then from the right triangle PBQ, we have

$$\begin{aligned} PQ^2 &= PB^2 + BQ^2 \\ &= AB^2 + PA^2 + BQ^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

So, the distance between point $P(x_1, y_1, z_1)$ and point $Q(x_2, y_2, z_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Exercise 3.1

1. Find the equation of the plane containing the point $(1, -2, 3)$ and parallel to
   (a) xy-plane (b) yz-plane (c) zx-plane.
2. Find the equation of the line through the point $(2, 3, -4)$ and perpendicular to
   (a) xy-plane (b) yz-plane (c) zx-plane.
   Find the point of intersection of the line and plane.
3. Find the distance between the points $(2, -3, 5)$ and $(7, 5, -2)$.
4. Show that the points $(-1, 2, 5), (1, 1, 6)$ and $(0, 5, 6)$ form a right triangle.
5. Show that the points $(1, -1, 2), (3, -2, 3)$ and $(5, -3, 4)$ are collinear.

3.2 Lines

Directed values of a line segment: For a line segment PQ, **directed vlaues** $\langle l, m, n\rangle$ of PQ, where P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2), is defined by

$$\boxed{\langle PQ\rangle = \langle l, m, n\rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle}$$

Then the length of the segment PQ is

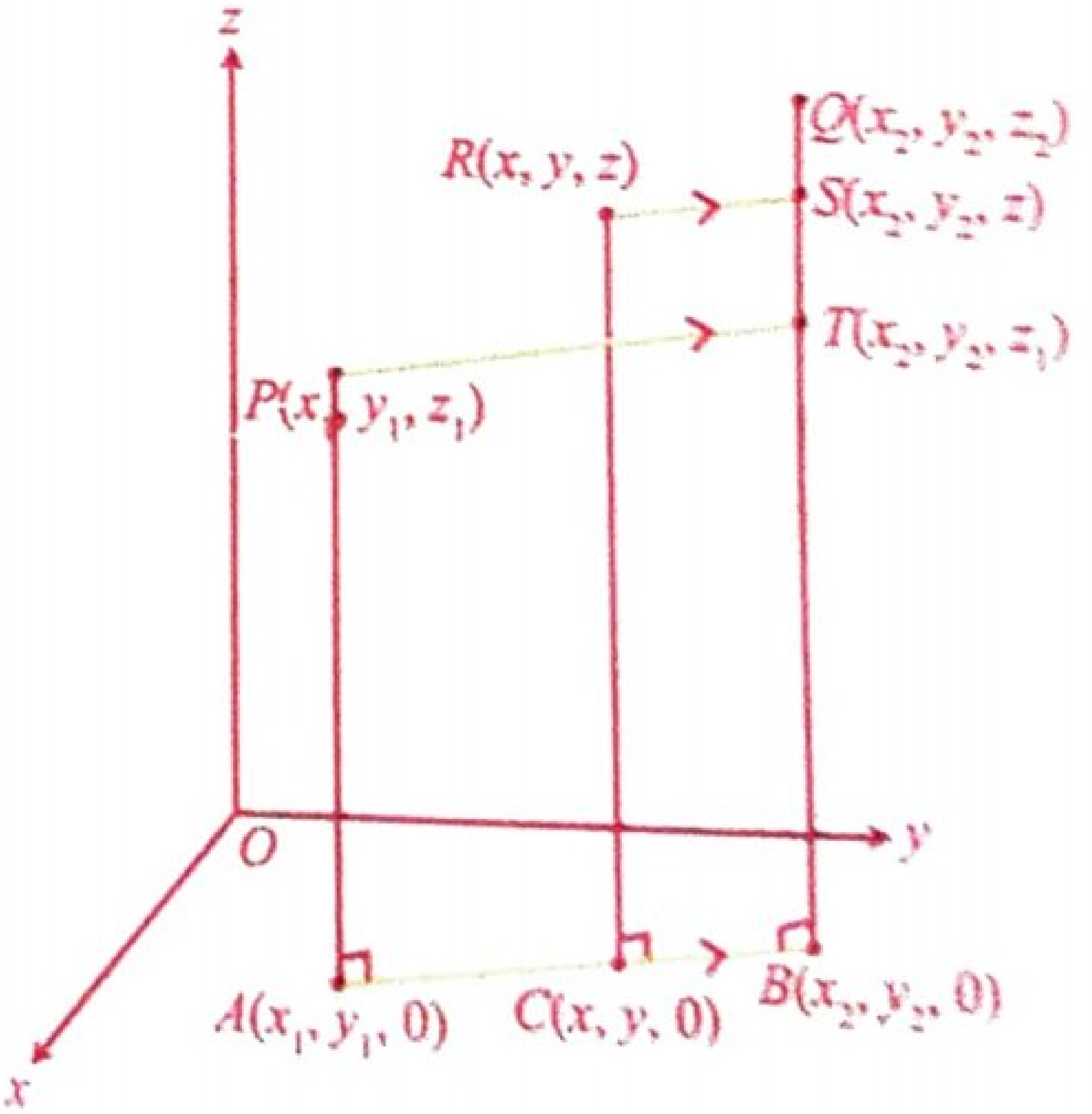
$$\boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{l^2 + m^2 + n^2}}$$

Real numbers and points on the line: For every real number k, there is a point R on the line PQ with respect to point P and k, and vice versa.

Case 1. $0 \leq k \leq 1$
Point R is between P and Q such that $PR = kPQ$.

Draw PA, RC and QB perpendicular to xy-plane: AB is **projection** of PQ on xy-plane.
Draw $PT \parallel AB$ and $RS \parallel AB$. Then

$$\begin{aligned} \frac{TS}{TQ} &= \frac{PR}{PQ} \\ \frac{z - z_1}{z_2 - z_1} &= k \\ z &= z_1 + k(z_2 - z_1) \\ z &= z_1 + kn \end{aligned}$$



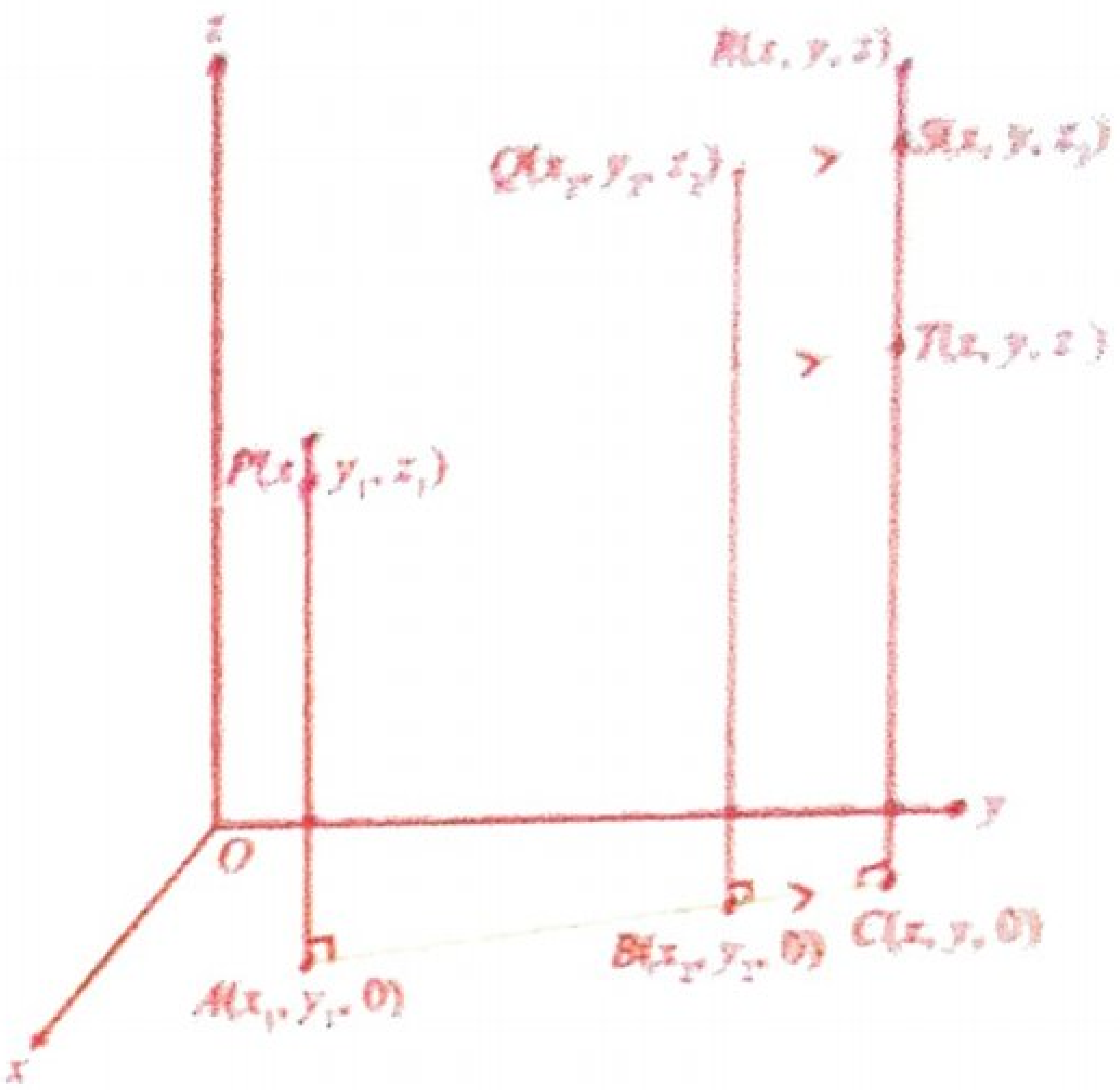
Similarly, by drawing the projection of PQ on yz-plane, we have

$$\begin{aligned} \frac{x - x_1}{x_2 - x_1} &= k \\ x &= x_1 + k(x_2 - x_1) \\ x &= x_1 + kl \end{aligned}$$

and by drawing the projection of PQ on zx-plane, we have

$$\begin{aligned} \frac{y - y_1}{y_2 - y_1} &= k \\ y &= y_1 + k(y_2 - y_1) \\ y &= y_1 + km \end{aligned}$$

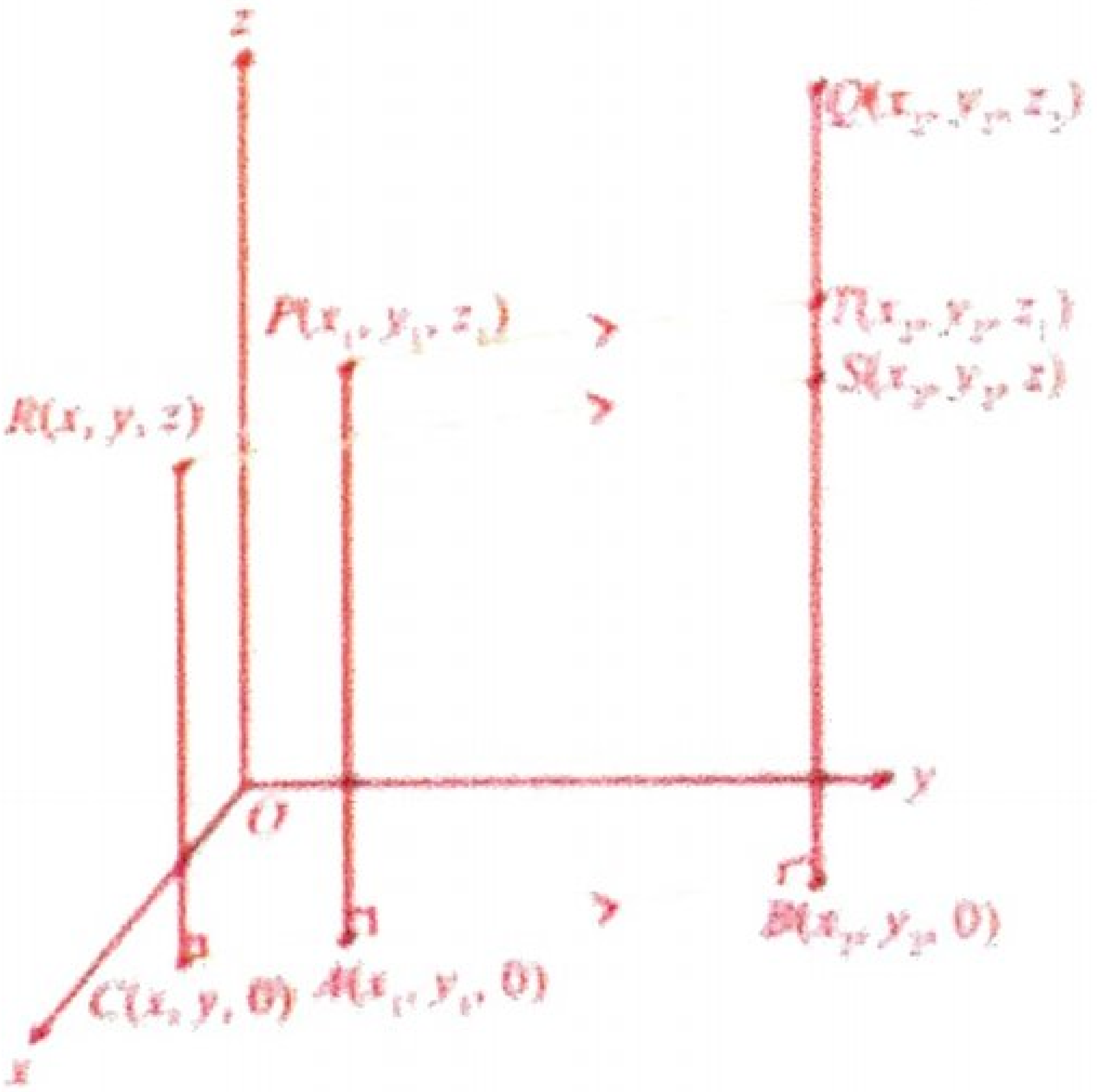
se 2. $k > 1$
int R is after Q such that $PR = kPQ$.


z
$R(x, y, z)$
$Q(x_2, y_2, z_2)$
$S(x, y, z_2)$
$T(x, y, z_1)$
$P(x_1, y_1, z_1)$
O
y
$B(x_2, y_2, 0)$
$C(x, y, 0)$
$A(x_1, y_1, 0)$
x

aw $PT \parallel AC$ and $QS \parallel AC$. Then we
ve

$$\frac{TR}{TS} = \frac{PR}{PQ}$$
$$\frac{z - z_1}{z_2 - z_1} = k$$
$$z = z_1 + k(z_2 - z_1)$$
$$z = z_1 + kn$$

se 3. $k < 0$
int R is before P such that $RP = -kPQ$.


z
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
$T(x_2, y_2, z_1)$
$S(x_2, y_2, z)$
$R(x, y, z)$
O
y
$B(x_2, y_2, 0)$
$C(x, y, 0)$
$A(x_1, y_1, 0)$
x

aw $PT \parallel CB$ and $RS \parallel CB$. Then we
ve

$$\frac{ST}{TQ} = \frac{RP}{PQ}$$
$$\frac{z_1 - z}{z_2 - z_1} = -k$$
$$z = z_1 + k(z_2 - z_1)$$
$$z = z_1 + kn$$

Similarly, we have for Case 2 and Case 3,

$$\frac{x - x_1}{x_2 - x_1} = k \qquad \frac{y - y_1}{y_2 - y_1} = k$$
$$x = x_1 + k(x_2 - x_1) \qquad y = y_1 + k(y_2 - y_1)$$
$$x = x_1 + kl \qquad y = y_1 + km$$

Therefore the coordinates of point R on the line PQ with respect to the point
d a real number k are

$$\boxed{(x, y, z) = (x_1 + kl, y_1 + km, z_1 + kn)}$$

This equation is called **coordinate form** of the equation of line PQ and k is called a **parameter**.

In general, if a segment of a line through (x_1, y_1, z_1) has directed values $\langle l, m$ which are not equal to 0, then the equation of the line can be written as

$$\boxed{\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}}$$

Since

$$\begin{aligned}\langle PR\rangle &= \langle x_1 + kl - x_1, y_1 + km - y_1, z_1 + kn - z_1\rangle\\ &= \langle kl, km, kn\rangle,\end{aligned}$$

and R is any point on the line PQ with $\langle PQ\rangle = \langle l, m, n\rangle$, we may define directed va of the line PQ as $\langle kl, km, kn\rangle$ where k is a real number.

Example 2.

Given $P(1,2,3)$ and $Q(3,6,5)$, find the coordinates of point $R(x,y,z)$ on the line with respect to the point P and the following parameters.

(a) $k = \frac{1}{2}$ (b) $k = 2$ (c) $k = -2$

Solution

Given $P(1,2,3)$ and $Q(3,6,5)$, we have $\langle PQ\rangle = \langle l, m, n\rangle = \langle 2,4,2\rangle$.

(a) $k = \frac{1}{2}$: $(x,y,z) = (1 + \frac{1}{2}(2), 2 + \frac{1}{2}(4), 3 + \frac{1}{2}(2)) = (2,4,4)$

(b) $k = 2$: $(x,y,z) = (1 + 2(2), 2 + 2(4), 3 + 2(2)) = (5,10,7)$

(c) $k = -2$: $(x,y,z) = (1 + (-2)(2), 2 + (-2)(4), 3 + (-2)(2)) = (-3,-6,-1$

Example 3.

Given $P(-1,2,3)$ and $Q(3,5,-2)$, determine whether or not the following points on the line PQ. If the point is on the line PQ, find the corresponding parameter respect to the point P.

(a) $(1, \frac{7}{2}, \frac{1}{2})$ (b) $(7,8,-7)$ (c) $(-5,-1,8)$ (d) $(7,8,-2$

Solution

Given $P(-1,2,3)$ and $Q(3,5,-2)$, we have $\langle PQ\rangle = \langle l, m, n\rangle = \langle 4,3,-5\rangle$. The equation of the line PQ is

$$\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}.$$

(a) If $(x,y,z) = (1, \frac{7}{2}, \frac{1}{2})$, then

$$\frac{x+1}{4} = \frac{1+1}{4} = \frac{1}{2}$$
$$\frac{y-2}{3} = \frac{\frac{7}{2}-2}{3} = \frac{1}{2}$$
$$\frac{z-3}{-5} = \frac{\frac{1}{2}-3}{-5} = \frac{1}{2}$$

and so $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $(x,y,z) = (1, \frac{7}{2}, \frac{1}{2})$.
Therefore the point $(1, \frac{7}{2}, \frac{1}{2})$ is on the line PQ, corresponding parameter is $\frac{1}{2}$.

(b) If $(x,y,z) = (7,8,-7)$, then

$$\frac{x+1}{4} = \frac{7+1}{4} = 2$$
$$\frac{y-2}{3} = \frac{8-2}{3} = 2$$
$$\frac{z-3}{-5} = \frac{-7-3}{-5} = 2$$

and so $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $(x,y,z) = (7,8,-7)$.
Therefore the point $(7,8,-7)$ is on the line PQ, corresponding parameter is 2.

c) If $(x,y,z) = (-5,-1,8)$, then

$$\frac{x+1}{4} = \frac{-5+1}{4} = -1$$
$$\frac{y-2}{3} = \frac{-1-2}{3} = -1$$
$$\frac{z-3}{-5} = \frac{8-3}{-5} = -1$$

nd so $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $(x,y,z) = (-5,-1,8)$.
herefore the point $(-5,-1,8)$ is on the line PQ, corresponding parameter is -1.

d) If $(x,y,z) = (7,8,-2)$, then

$$\frac{x+1}{4} = \frac{7+1}{4} = 2$$
$$\frac{y-2}{3} = \frac{8-2}{3} = 2$$
$$\frac{z-3}{-5} = \frac{-2-3}{-5} = 1$$

ad so $\frac{x+1}{4} = \frac{y-2}{3} \neq \frac{z-3}{-5}$ for $(x, y, z) = (7, 8, -2)$.
herefore the point $(7, 8, -2)$ is not on the line PQ.

xample 4.

iven $P(2, 1, 3)$ and $Q(6, -5, 3)$, determine whether or not the following points a
e line PQ. If the point is on the line PQ, find the corresponding paramete
spect to the point P.

) $(4, -2, 3)$ (b) $(-2, 7, 3)$ (c) $(10, -11, 3)$ (d) $(1, 1, 3$

olution

iven $P(2, 1, 3)$ and $Q(6, -5, 3)$, we have $\langle PQ \rangle = \langle l, m, n \rangle = \langle 4, -6, 0 \rangle$
hen the coordinates of the point (x, y, z) on the line PQ are

$$(x, y, z) = (2 + 4k, 1 - 6k, 3)$$

his means that the line PQ is on the plane $z = 3$.

) If $(x, y, z) = (4, -2, 3)$, then

$$(4, -2, 3) = (2 + 4k, 1 - 6k, 3),$$

e have $k = \frac{1}{2}$. Therefore the point $(4, -2, 3)$ is on the line PQ with corresp
arameter $\frac{1}{2}$.

) If $(x, y, z) = (-2, 7, 3)$, then

$$(-2, 7, 3) = (2 + 4k, 1 - 6k, 3),$$

e have $k = -1$. Therefore the point $(-2, 7, 3)$ is on the line PQ with corresp
arameter -1.

) If $(x, y, z) = (10, -11, 3)$, then

$$(10, -11, 3) = (2 + 4k, 1 - 6k, 3),$$

e have $k = 2$. Therefore the point $(10, -11, 3)$ is on the line PQ with corresp
arameter 2.

d) If $(x, y, z) = (1, 1, 3)$, then

$$(1, 1, 3) = (2 + 4k, 1 - 6k, 3),$$

here is no value k that satisfies this condition. Therefore the point $(1, 1, 3)$ is
he line PQ.

Exercise 3.2

1. Given $P(3,1,5)$ and $Q(-3,7,-2)$, find the coordinates of point $R(x,y,z)$ on the line PQ with respect to the point P and the following parameters.

   (a) $k = \frac{1}{2}$ (b) $k = 3$ (c) $k = -2$

2. Given $P(-2,1,3)$ and $Q(4,4,-3)$, determine whether or not the following points are on the line PQ. If the point is on the line PQ, find the corresponding parameter with respect to the point P.

   (a) $(6,3,-6)$ (b) $(6,5,-5)$ (c) $(-4,0,5)$ (d) $(7,8,-2)$

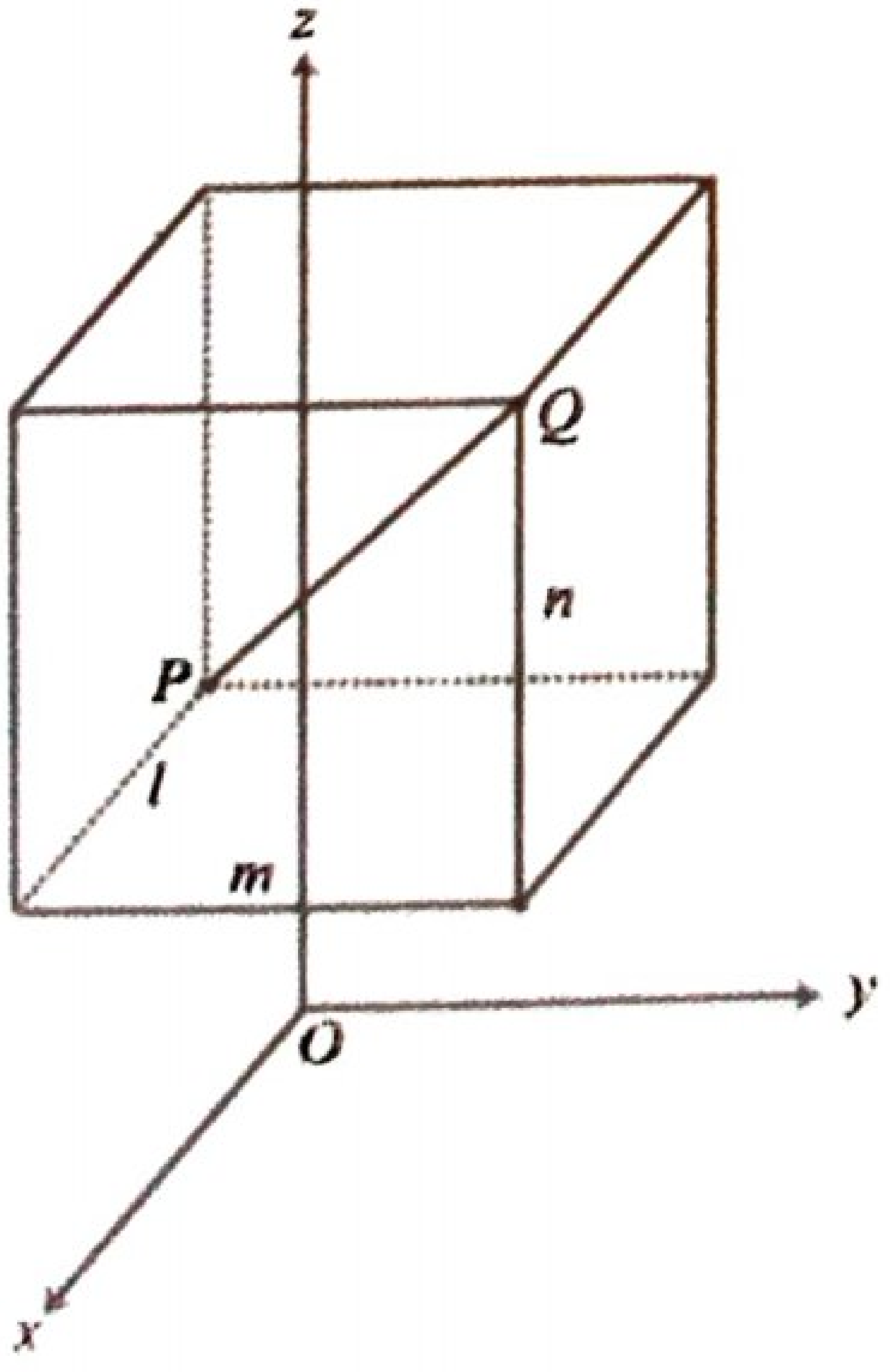
3. Given $P(3,2,-1)$ and $Q(4,2,5)$, determine whether or not the following points are on the line PQ. If the point is on the line PQ, find the corresponding real number with respect to the point P.

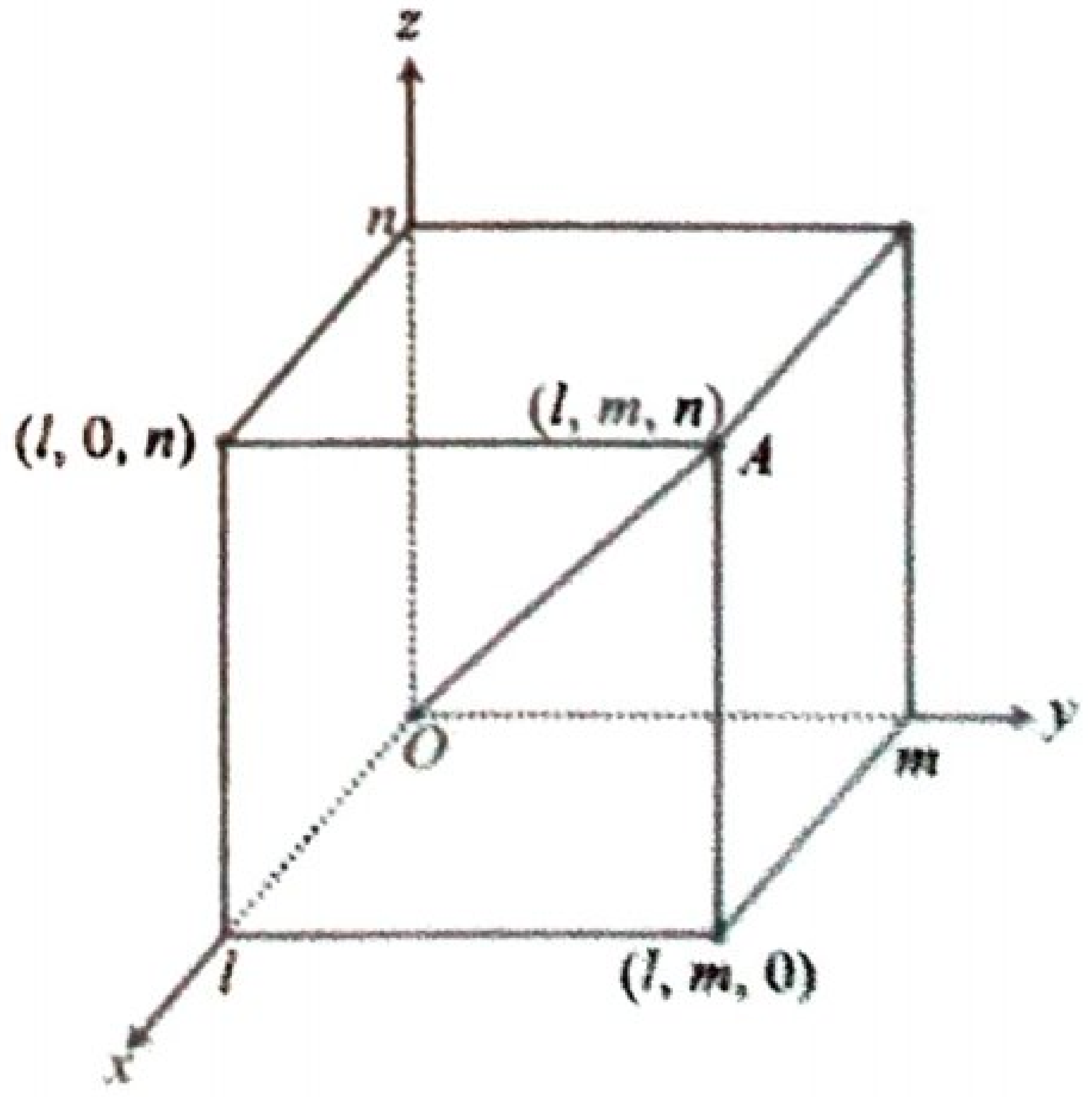
   (a) $(5,2,11)$ (b) $(2,2,-7)$ (c) $(\frac{7}{2},2,2)$ (d) $(6,2,10)$

4. Find the points of intersection of the line joining the two points $(2,4,5)$ and $(3,5,-4)$ with the following planes.
   (a) xy-plane (b) yz-plane (c) zx-plane.

3.3 Parallel, Skew and Perpendicular Lines

Parallel lines:





If P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2) with directed values

$$\langle PQ\rangle = \langle l, m, n\rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle,$$

then segment PQ and segment OA where $A = (l, m, n)$ are parallel and $PQ = OA$ as shown in the given figure.

So line PQ and line OA are parallel if they have same directed values. But if directed values of a line are given by $\langle l, m, n\rangle$, then $\langle kl, km, kn\rangle$ are also directed values of the line. Therefore line PQ and line OA are parallel if their directed values are multiples of each other by a real number k. Any line parallel to the line OA is also parallel to the line PQ, so that

Two lines are parallel if and only if their directed values are multiples of each other by some real number.

For example, if line PQ has a directed values $\langle 2, 3, 5\rangle$ and line RS has a directed values $\langle 4, 6, 10\rangle$ then $PQ \parallel RS$.

Skew lines: In space, there are pairs of lines that are neither parallel nor intersect. These pairs of lines are called **skew** lines.

Example 5.

Given $P(2, 1, 3)$, $Q(6, -5, 4)$, $R(2, 3, 4)$ and $S(-1, 5, 4)$, determine whether the lines PQ and RS are parallel or skew or intersect.

Solution

For $P(2, 1, 3)$ and $Q(6, -5, 4)$, $\langle PQ\rangle = \langle 4, -6, 1\rangle$.
For $R(2, 3, 4)$ and $S(-1, 5, 1)$, $\langle RS\rangle = \langle -3, 2, -3\rangle$.
Since $\frac{4}{-3} \neq \frac{-6}{2} \neq \frac{1}{-3}$, directed values of $\langle PQ\rangle$ are not multiple of $\langle RS\rangle$. So two lines are not parallel.

If a point (x, y, z) is on the lines PQ and RS, then

$$x = 2 + 4s \qquad x = 2 - 3t$$
$$y = 1 - 6s \qquad y = 3 + 2t$$
$$z = 3 + s \qquad z = 4 - 3t$$

for real numbers s and t. Then we have the following system of three equations

$$\begin{aligned} 2 + 4s &= 2 - 3t \\ 1 - 6s &= 3 + 2t \\ 3 + s &= 4 - 3t \end{aligned}$$

Solving first two of these equations, we have $s = -\frac{3}{5}$ and $t = \frac{4}{5}$. But

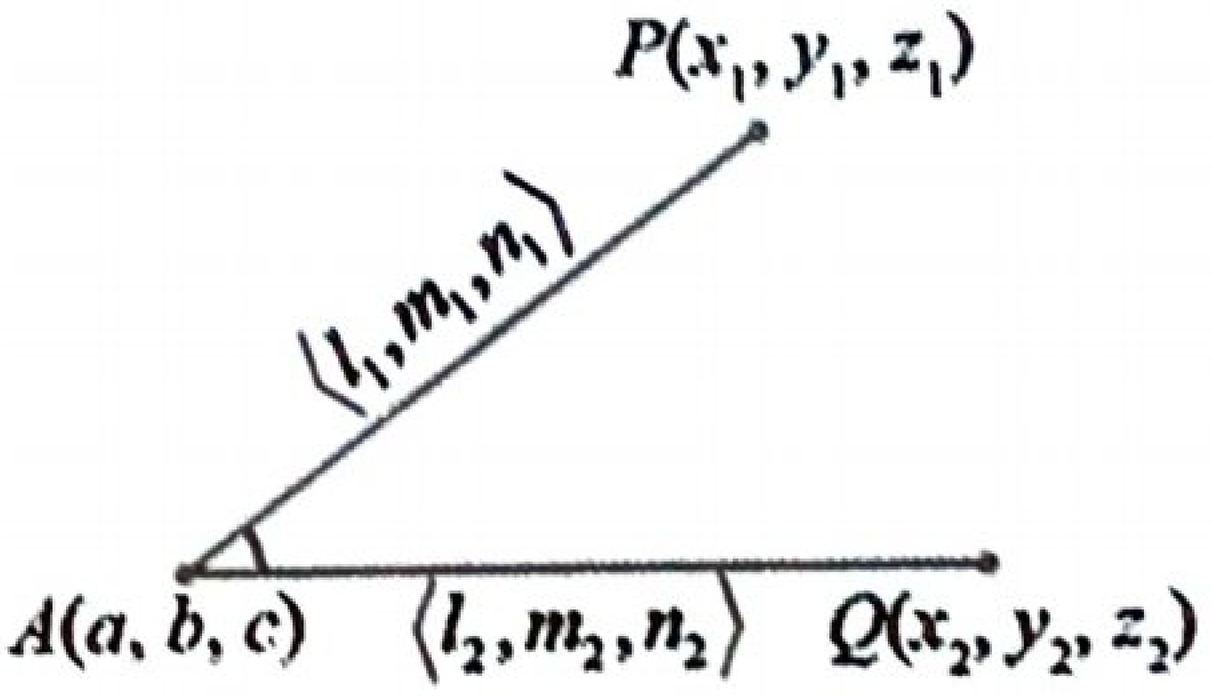
$$3 + (-\frac{3}{5}) \neq 4 - 3(\frac{4}{5}),$$

these values of s and t do not satisfy the last equation. So the system of equations has no solution, and hence the given lines do not intersect.

Therefore the given lines are skew.

Finding the measure of $\angle PAQ$: Consider $P(x_1, y_1, z_1)$, $A(a, b, c)$ and $Q(x_2, y_2, z_2)$.

$$\begin{aligned}
\langle AP \rangle &= \langle l_1, m_1, n_1 \rangle = \langle x_1 - a, y_1 - b, z_1 - c \rangle \\
\langle AQ \rangle &= \langle l_2, m_2, n_2 \rangle = \langle x_2 - a, y_2 - b, z_2 - c \rangle \\
\langle PQ \rangle &= \langle l_3, m_3, n_3 \rangle \\
&= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\
&= \langle l_2 - l_1, m_2 - m_1, z_2 - z_1 \rangle
\end{aligned}$$


$P(x_1, y_1, z_1)$
$\langle l_1, m_1, n_1 \rangle$
$A(a, b, c)$
$\langle l_2, m_2, n_2 \rangle$
$Q(x_2, y_2, z_2)$

$$\begin{aligned}
& AP^2 + AQ^2 - PQ^2 \\
&= l_1^2 + m_1^2 + n_1^2 + l_2^2 + m_2^2 + n_2^2 - (l_2 - l_1)^2 - (m_2 - m_1)^2 - (n_2 - n_1)^2 \\
&= 2(l_1 l_2 + m_1 m_2 + n_1 n_2)
\end{aligned}$$

By the law of cosines,

$$\begin{aligned}
\cos \angle PAQ &= \frac{AP^2 + AQ^2 - PQ^2}{2 \cdot AP \cdot AQ} \\
&= \frac{2(l_1 l_2 + m_1 m_2 + n_1 n_2)}{2 \cdot AP \cdot AQ} \\
&= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}
\end{aligned}$$

If $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, then $\cos \angle PAQ = 0$, and hence $\angle PAQ = 90°$.

Perpendicular lines: Two lines are perpendicular if and only if they are intersect and

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

for any directed values $\langle l_1, m_1, n_1 \rangle$ and $\langle l_2, m_2, n_2 \rangle$ of the lines.

Example 6.

Given $P(0, 0, 1)$, $Q(3, 6, 4)$, $R(0, 3, 1)$ and $S(3, 0, 4)$, show that the lines PQ and RS are perpendicular.

Solution

Given $P(0,0,1)$ and $Q(3,6,4)$, we have $\langle PQ\rangle = \langle 3,6,3\rangle$.
Given $R(0,3,1)$ and $S(3,0,4)$, we have $\langle RS\rangle = \langle 3,-3,3\rangle$.
And we have $(3)(3)+(6)(-3)+(3)(3)=0$.

If a point (x,y,z) is on the lines PQ and RS, then

$$\begin{aligned} x &= 0+3s & \qquad x &= 0+3t \\ y &= 0+6s & \qquad y &= 3-3t \\ z &= 1+3s & \qquad z &= 1+3t \end{aligned}$$

for real numbers s and t. Then we have the following system of three equations

$$\begin{aligned} 3s &= 3t \\ 6s &= 3-3t \\ 1+3s &= 1+3t \end{aligned}$$

Solving these equations, we have $s=\dfrac{1}{3}$ and $t=\dfrac{1}{3}$. The point of intersection is $(1, 2$
and two lines intersect. Therefore two lines are perpendicular.

Example 7.

Find the equation of the line passing through the point $(-4,7,-3)$ and perpendic
to the line

$$(x,y,z) = (3+2k, -1+3k, 1-k).$$

Find also the point of intersection of two lines.

Solution

Directed values of the given line are

$$\langle 2,3,-1\rangle.$$

Directed values of the required line are

$$\langle -4-(3+2k), 7-(-1+3k), -3-(1-k)\rangle = \langle -7-2k, 8-3k, -4+k$$

for some real number k.

If two lines are perpendicular, then

$$\begin{aligned} 2(-7-2k)+3(8-3k)+(-1)(-4+k) &= 0 \\ -14-4k+24-9k+4-k &= 0 \\ 14k &= 14 \\ k &= 1 \end{aligned}$$

So directed values of the required line are $\langle -9, 5, -3\rangle$ and the equation of the line is

$$(x, y, z) = (-4 - 9t, 7 + 5t, -3 - 3t)$$

The point of intersection is

$$(x, y, z) = (5, 2, 0).$$

Exercise 3.3

1. Find $\cos \angle PAQ$ for the followings.
   (a) $P(1,2,-1)$, $A(-2,1,5)$, $Q(2,-1,0)$
   (b) $P(0,2,-3)$, $A(2,-1,5)$, $Q(-2,3,-1)$

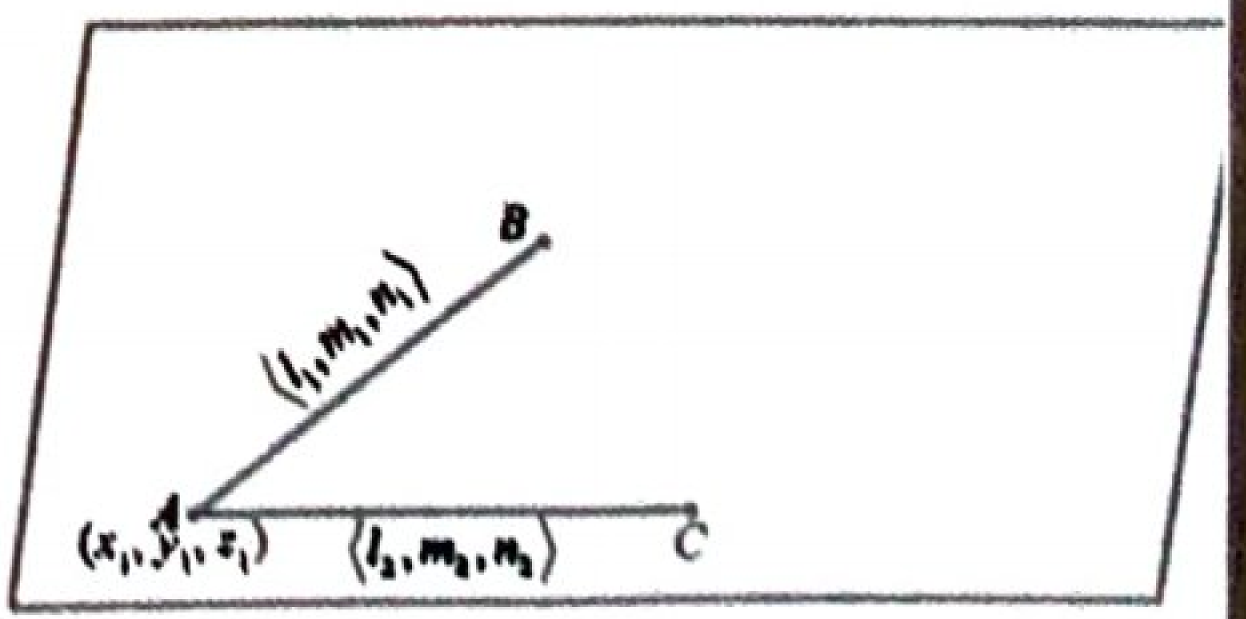
2. Determine whether the lines PQ and RS are parallel or skew or intersect. If PQ and RS intersect, are they perpendicular?
   (a) $P(1,2,3), Q(4,5,6), R(-2,3,5), S(4,9,11)$
   (b) $P(3,-1,-3), Q(2,-3,1), R(3,-2,5), S(-1,-2,1)$
   (c) $P(4,-2,5), Q(-2,6,1), R(-1,1,4), S(3,3,2)$
   (d) $P(-3,-1,6), Q(-1,3,0), R(0,6,7), S(-4,-4,-1)$

3. Find the equation of the line passing through the point $(8,-1,-10)$ and perpendicular to the line

$$(x, y, z) = (1 + 2k, 2 - k, 3 - 7k).$$

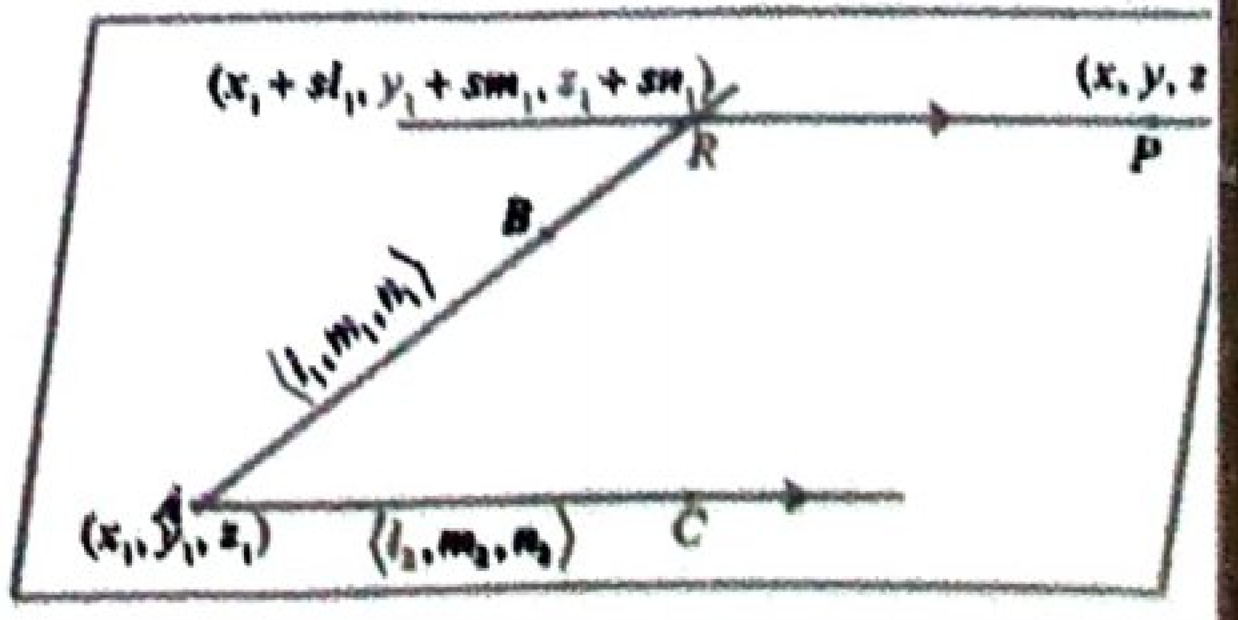
   Find also the point of intersection of two lines.

3.4 Planes


B
$\langle l_1, m_1, n_1\rangle$
A
(x_1, y_1, z_1)
$\langle l_2, m_2, n_2\rangle$
C

A plane is determined by three points which are not on the same line.

Let $P(x,y,z)$ be any point on the plane through A, B, C. Since A, B, C are not on the same line, line segment joining any two points will intersect each other. Let AB intersect AC at A. Draw a line through P parallel to AC. This line will meet AB at $R(x_1 + sl_1, y_1 + sm_1, z_1 + sn_1)$ for some parameter s.

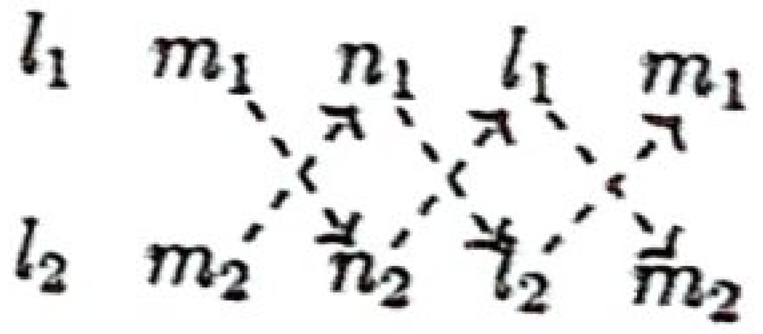

$(x_1 + sl_1, y_1 + sm_1, z_1 + sn_1)$
$(x, y, z$
R
P
B
$\langle l_1, m_1, n_1\rangle$
A
(x_1, y_1, z_1)
$\langle l_2, m_2, n_2\rangle$
C

As shown in the given figure, coordinates of any point (x, y, z) on the plane are

$$x = x_1 + sl_1 + tl_2$$
$$y = y_1 + sm_1 + tm_2$$
$$z = z_1 + sn_1 + tn_2, \quad \text{for some parameter } t.$$

Let

$$a = m_1n_2 - m_2n_1$$
$$b = n_1l_2 - n_2l_1$$
$$c = l_1m_2 - l_2m_1$$



Then

$$al_1 + bm_1 + cn_1 = 0$$
$$\text{and} \quad al_2 + bm_2 + cn_2 = 0.$$

Thus we have the following plane equation:

$$ax + by + cz = ax_1 + by_1 + cz_1.$$

Let $d = ax_1 + by_1 + cz_1$, then we have the **Cartesian form** of the plane equation as

$$\boxed{ax + by + cz = d}$$

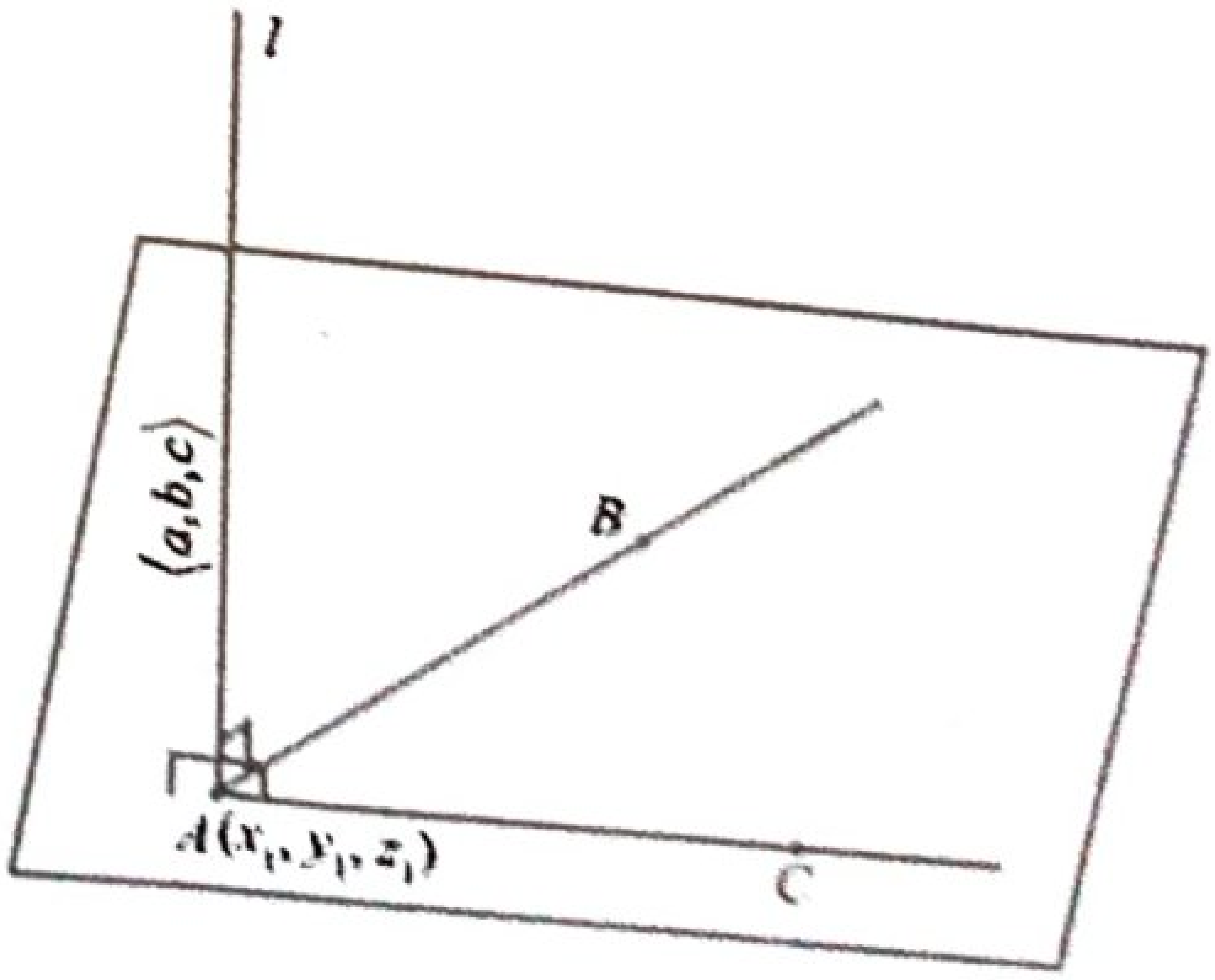
Since

$$al_1 + bm_1 + cn_1 = 0$$
$$\text{and} \quad al_2 + bm_2 + cn_2 = 0,$$

the line l with equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

is perpendicular to both of the lines AB and AC, so the line l is perpendicular to the plane ABC. Hence any line with directed values $\langle ka, kb, kc\rangle$, for some parameter k, is perpendicular to the plane ABC.



Example 8.

Find the equation of the plane containing $A(1, 0, 1)$, $B(3, 6, 4)$ and $C(-2, 3, 1)$.

Solution

Given $A(1, 0, 1)$ and $B(3, 6, 4)$, we have $\langle AB\rangle = \langle l_1, m_1, n_1\rangle = \langle 2, 6, 3\rangle$.

Given $A(1,0,1)$ and $C(-2,3,1)$, we have $\langle AC\rangle = \langle l_2, m_2, n_2\rangle = \langle -3, 3, 0\rangle$

$$\begin{aligned} a &= m_1n_2 - m_2n_1 = 6(0) - 3(3) = -9 \\ b &= n_1l_2 - n_2l_1 = 3(-3) - 0(2) = -9 \\ c &= l_1m_2 - l_2m_1 = 2(3) - (-3)(6) = 24 \end{aligned}$$

and

$$d = ax_1 + by_1 + cz_1 = -9(1) - 9(0) + 24(1) = 15$$

Therefore the equation of the plane is

$$-9x - 9y + 24z = 15.$$

Note that we can use any point on the plane to calculate the value of d.

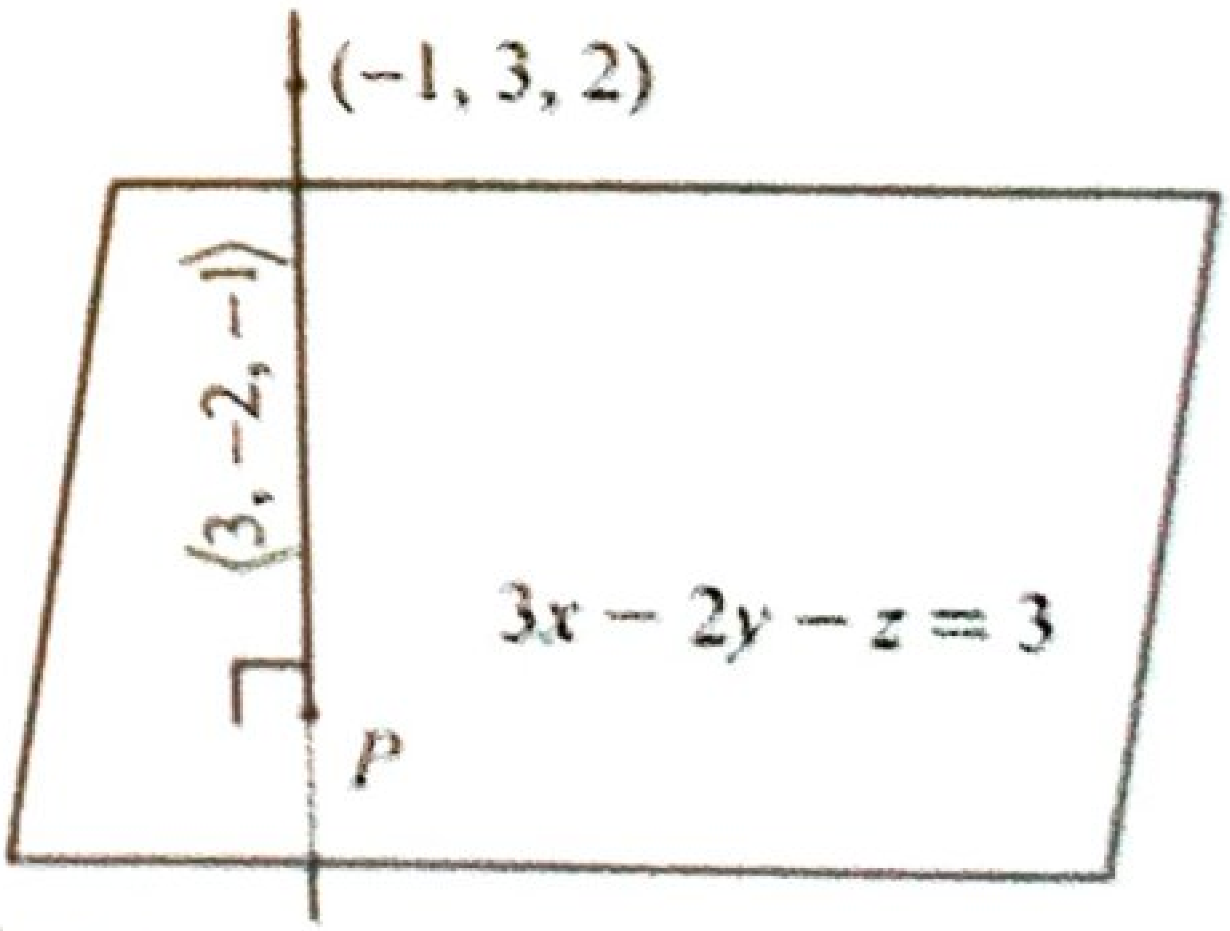
Example 9.

Find the equation of the line that passes through the point $(-1, 3, 2)$ and perpendicular to the plane $3x - 2y - z = 3$. Find the point of intersection of the line and the given plane.

Solution

Directed values of the line perpendicular to the plane $3x - 2y - z = 3$ are $\langle 3, -2, -1\rangle$. Then equation of the line is

$$\begin{aligned} \frac{x-(-1)}{3} &= \frac{y-3}{-2} = \frac{z-2}{-1} \\ \frac{x+1}{3} &= \frac{y-3}{-2} = \frac{z-2}{-1} \end{aligned}$$



So coordinates of the points on the line are of the form

$$\begin{aligned} x &= -1 + 3s \\ y &= 3 + (-2)s = 3 - 2s \\ z &= 2 + (-1)s = 2 - s \end{aligned}$$

If one of these points P is on the plane, then

$$\begin{aligned} 3(-1 + 3s) - 2(3 - 2s) - (2 - s) &= 3 \\ -3 + 9s - 6 + 4s - 2 + s &= 3 \\ 14s &= 14 \\ s &= 1 \end{aligned}$$

Therefore the point of intersection is $(2, 1, 1)$.

Example 10.

Find the equation of the plane containing the point $(-1, 3, 2)$ and parallel to the plane $3x - 2y - 3z = 2$.

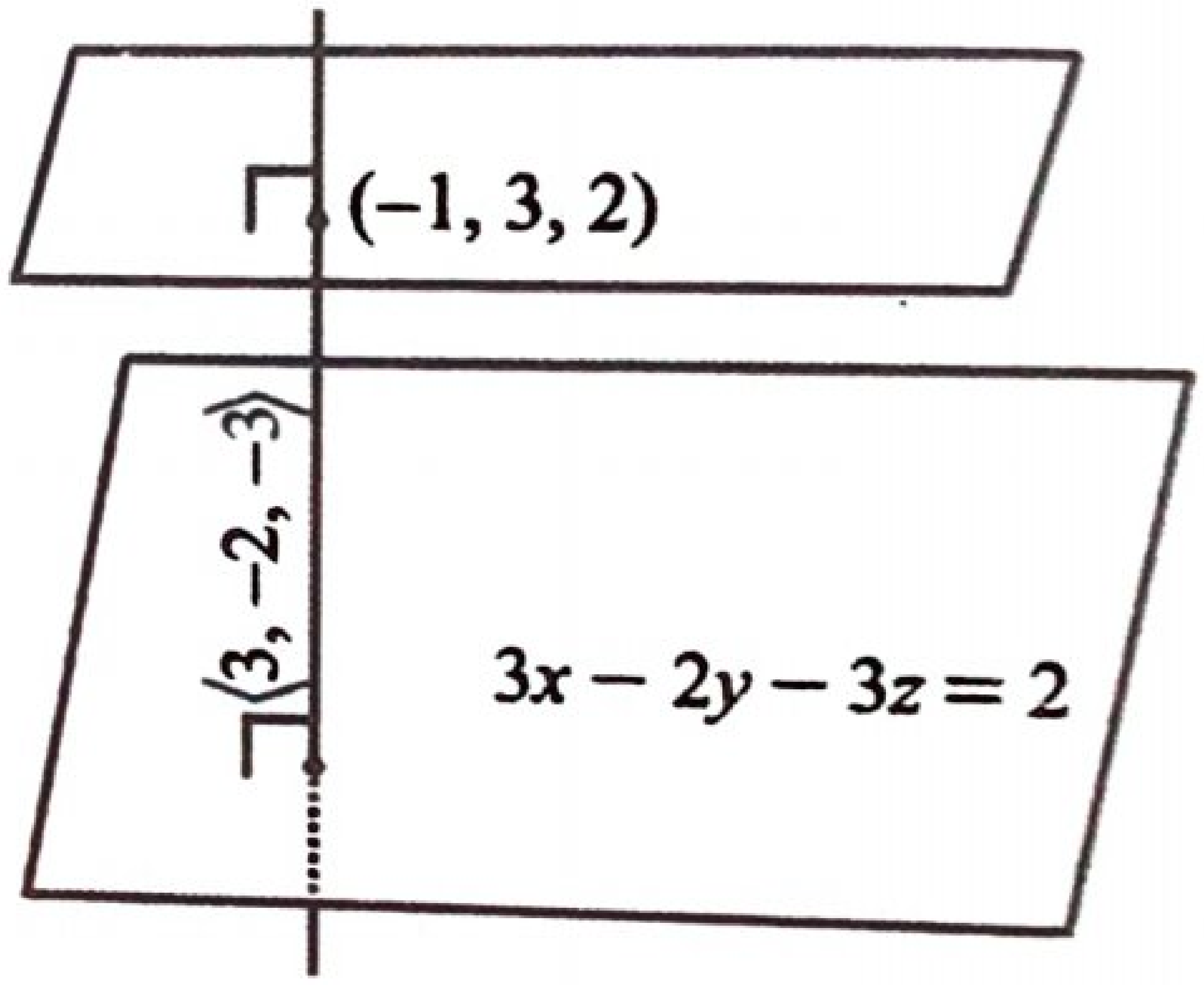
Solution
Directed values of the line perpendicular to the plane $3x - 2y - 3z = 2$ are $\langle 3, -2, -3 \rangle$. This line is also perpendicular to the required plane. So equation of the required plane is

$$3x - 2y - 3z = d$$

The point $(-1, 3, 2)$ is on this plane. So

$$3(-1) - 2(3) - 3(2) = d$$
$$d = -15$$

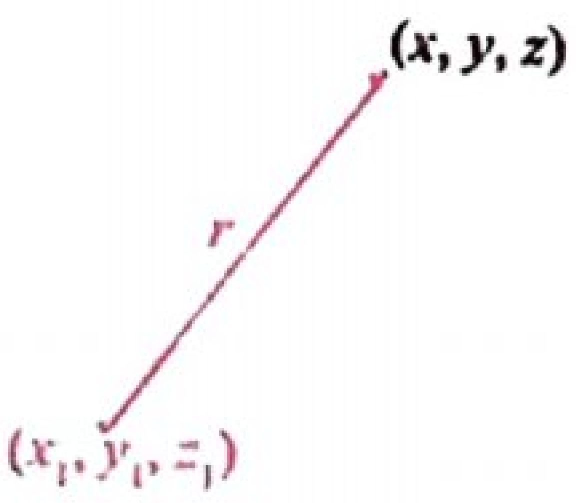
The equation of required plane is $3x - 2y - 3z = -15$.



Exercise 3.4

1. Find the equation of the palne containing
   (a) $A(2, -5, 4)$, $B(-5, 2, 4)$ and $C(-2, 3, -1)$
   (b) $A(4, 2, -3)$, $B(1, -2, 4)$ and $C(-1, 0, 3)$
2. Find the equation of the line passing through the point $(3, -2, -2)$ and perpendicular to the plane $-2x + 3y - z = 4$. Find the point of intersection of the line and the plane.
3. Find the equation of the plane containing the point $(2, 3, -1)$ and parallel to the plane $-2x + y + 3z = 6$.

3.5 Spheres



The distance between center (x_1, y_1, z_1) and any point (x, y, z) of a **sphere** is radius r.

$$\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}=r$$

Therefore the equation of the sphere with center (x_1, y_1, z_1) and radius r is

$$\boxed{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2}$$

Example 11.

Find the equation of the plane tangent to the sphere

$$(x-2)^2+(y-1)^2+(z+1)^2=14$$

at the point $(3, 4, 1)$.

Solution

Directed values of the line joining center $C(2, 1, -1)$ of the sphere and the given point $P(3, 4, 1)$ are $\langle CP\rangle = \langle 1, 3, 2\rangle$. Line CP is perpendicular to the tangent plane. Thus equation of the plane is

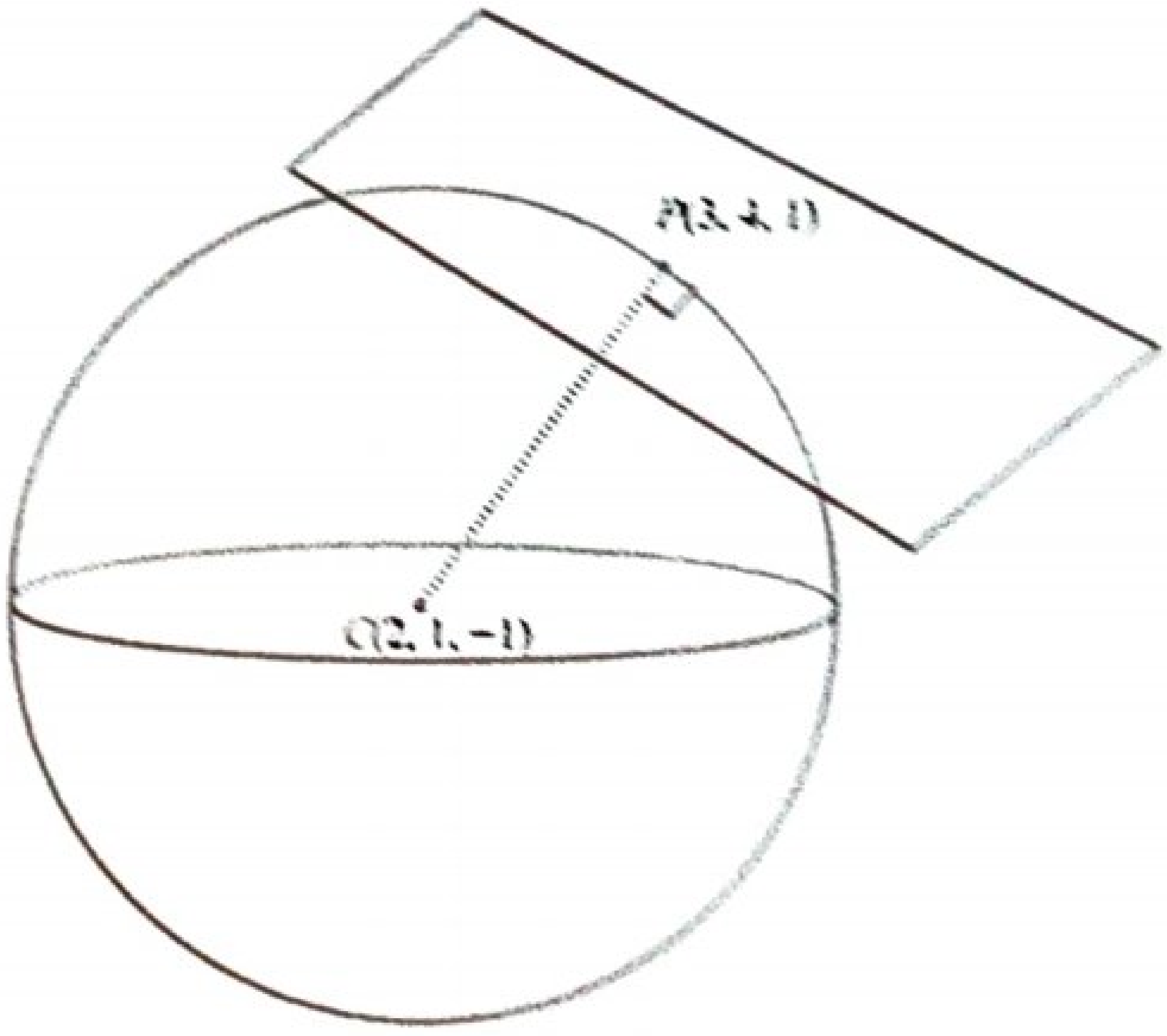
$$x+3y+2z=d.$$

Since $P(3, 4, 1)$ is on this plane, so we get

$$3+3(4)+2(1)=d$$
$$d=17$$

The equation of the plane is

$$x+3y+2z=17.$$



Example 12.

Find the equation of the sphere with center $(0, 1, 0)$ and touching the plane $x-2y+2z+5=0$.

Solution

The equation of the line that passes through the center $C(0, 1, 0)$ and perpendicular to the plane $x-2y+2z+5=0$ is

$$\frac{x-0}{1}=\frac{y-1}{-2}=\frac{z-0}{2}$$
$$\frac{x}{1}=\frac{y-1}{-2}=\frac{z}{2}$$

So coordinates of the points on the line are of the form

$$\begin{aligned} x &= s \\ y &= 1 + (-2)s = 1 - 2s \\ z &= 2s \end{aligned}$$

If one of these points P is on the plane, then

$$\begin{aligned} s - 2(1 - 2s) + 2(2s) + 5 &= 0 \\ s - 2 + 4s + 4s + 5 &= 0 \\ 9s &= -3 \\ s &= -\frac{1}{3} \end{aligned}$$

Therefore the point of intersection is $P(-\frac{1}{3}, \frac{5}{3}, -\frac{2}{3})$. Thus we have the radius

$$\begin{aligned} CP &= \sqrt{(-\frac{1}{3} - 0)^2 + (\frac{5}{3} - 1)^2 + (-\frac{2}{3} - 0)^2} \\ &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} \\ &= 1 \end{aligned}$$

Therefore the equation of the sphere is $x^2 + (y - 1)^2 + z^2 = 1$.

Example 13.

Find the equation of a sphere that passes through the points $(9, 0, 0)$, $(3, 13, 5)$ (11, 0, 10), given that its center lies on the yz-plane.

Solution

The equation of the sphere with center (x_1, y_1, z_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Distance from center (x_1, y_1, z_1) to each of the given points is

$$\begin{aligned} (9 - 0)^2 + (0 - y_1)^2 + (0 - z_1)^2 &= r^2 \quad (\text{ in } yz\text{-plane, } x_1 = 0) \\ 81 + y_1^2 + z_1^2 &= r^2 \\ (3 - 0)^2 + (13 - y_1)^2 + (5 - z_1)^2 &= r^2 \qquad (1) \\ 9 + (13 - y_1)^2 + (5 - z_1)^2 &= r^2 \\ (11 - 0)^2 + (0 - y_1)^2 + (10 - z_1)^2 &= r^2 \qquad (2) \\ 121 + y_1^2 + (10 - z_1)^2 &= r^2 \end{aligned}$$

(3)

From (1) and (3),

$$
\begin{aligned}
81 + y_1^2 + z_1^2 &= 121 + y_1^2 + (10 - z_1)^2 \\
81 + y_1^2 + z_1^2 &= 121 + y_1^2 + 100 - 20z_1 + z_1^2 \\
0 &= 40 + 100 - 20z_1 \\
z_1 &= 7
\end{aligned}
$$

From (1) and (2),

$$
\begin{aligned}
81 + y_1^2 + z_1^2 &= 9 + (13 - y_1)^2 + (5 - z_1)^2 \\
81 + y_1^2 + 7^2 &= 9 + 169 - 26y_1 + y_1^2 + (5 - 7)^2 \\
130 &= 182 - 26y_1 \\
y_1 &= 2
\end{aligned}
$$

Therefore $x_1 = 0, y_1 = 2$ and $z_1 = 7$ and substitute these values in equation (1), we get

$$r^2 = 134.$$

The equation of the sphere is

$$(x - 0)^2 + (y - 2)^2 + (z - 7)^2 = 134.$$

Exercise 3.5

1. Find the equation of sphere with center C and radius r.
   (a) $C(1, -2, 4)$, $r = 3$ (b) $C(2, 6, -3)$, $r = 2$ (c) $C(2, 3, 5)$, $r = 5$

2. Check whether the given point P lies inside, outside or on a sphere.
   (a) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(1, 1, 1)$
   (b) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(2, 1, 2)$
   (c) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(10, 10, 10)$

3. Find the equation of the sphere on the join of $(1, -1, 1)$ and $(-3, 4, 5)$ as diameter.

4. Find the equation of the plane tangent to the sphere

   $$(x + 2)^2 + (y - 1)^2 + (z + 3)^2 = 27$$

   at the point $(3, 2, -2)$.

5. Find the equation of the sphere with center $(6, -7, -3)$ and touching the plane $4x - 2y - z = 17$.

6. What is the equation of the sphere which passes through the points $(3, 0, 2)$, $(-1, 1, 1)$ and $(2, -5, 4)$ and whose center lies on the plane $2x + 3y + 4z = 6$?

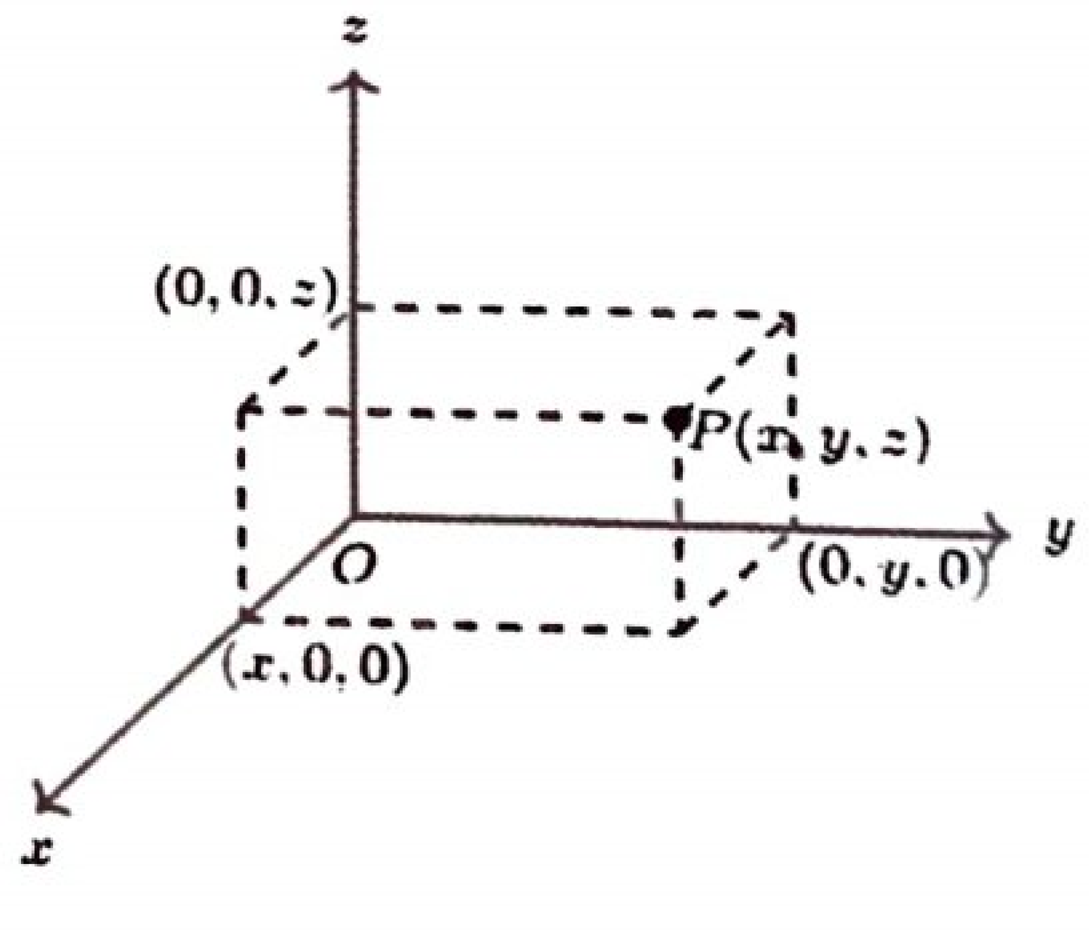
Chapter 4

Vector Algebra

We have learned vectors in two-dimensional rectangular coordinate system. In this chapter, we will extend the system to three dimensions and we will use vectors to calculate angles, distances and areas. We will also apply vectors to geometrical problems of lines and planes.

4.1 Vectors in Three Dimensions

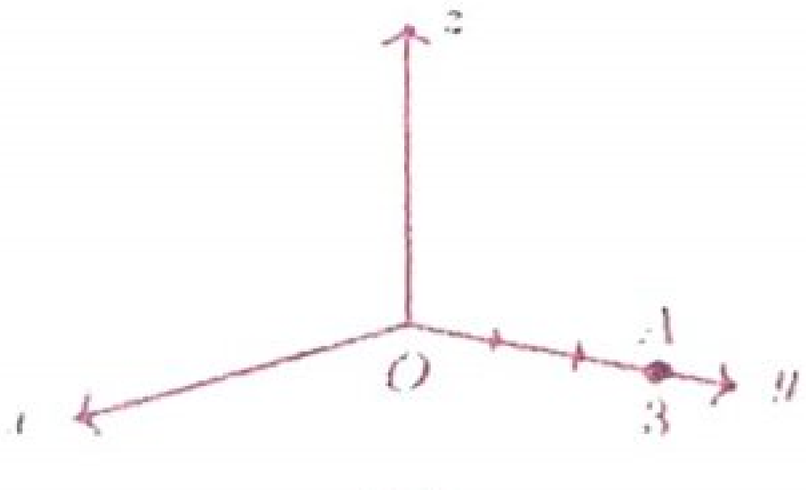
In the plane, each point is associated with an ordered pair of real numbers. In space, each point is associated with an ordered triple of real numbers. Through a fixed point, called the **origin O**, draw three mutually perpendicular lines: the x-axis, the y-axis and the z-axis. A point P in space is determined by an ordered triple (x, y, z) of real numbers as shown in the diagram. These numbers x, y, z are called the **coordinates** of P.



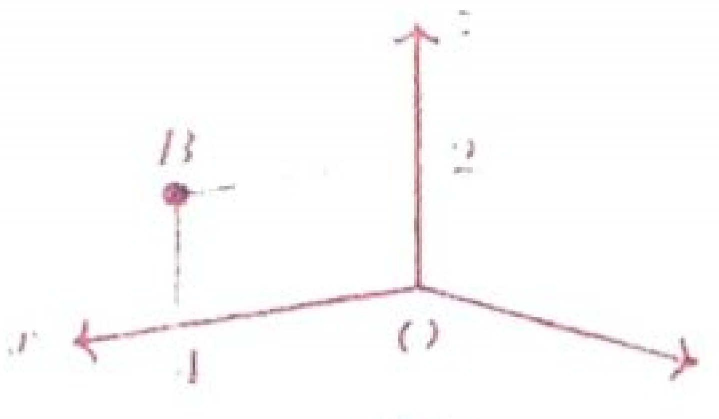
Example 1.

Illustrate the points (a) $A(0, 3, 0)$ (b) $B(4, 0, 2)$ (c) $C(-1, 2, 2)$

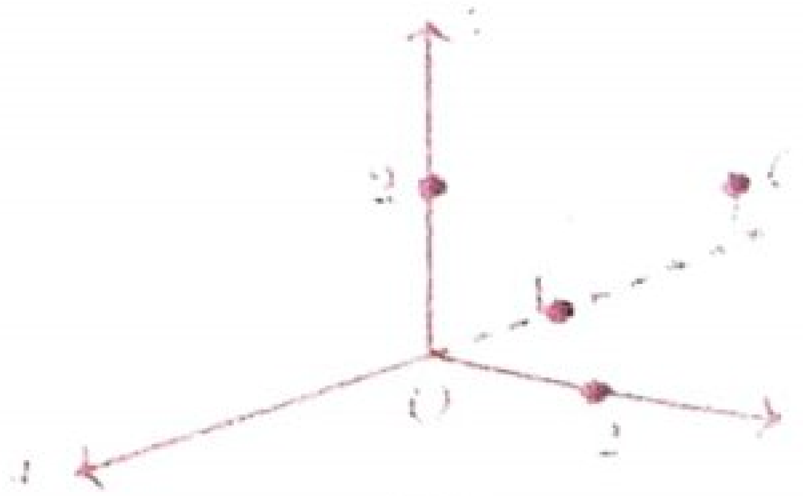
Solution



(a)



(b)



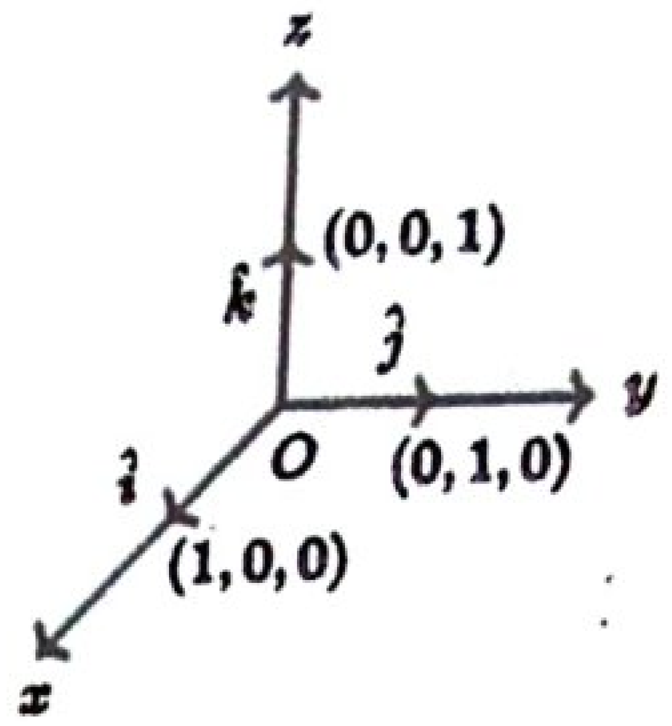
(c)

To represent vectors in space, we introduce the standard unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ where

$\hat{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ represents the unit vector in the positive x direction,

$\hat{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ represents the unit vector in the positive y direction and

$\hat{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ represents the unit vector in the positive z direction.



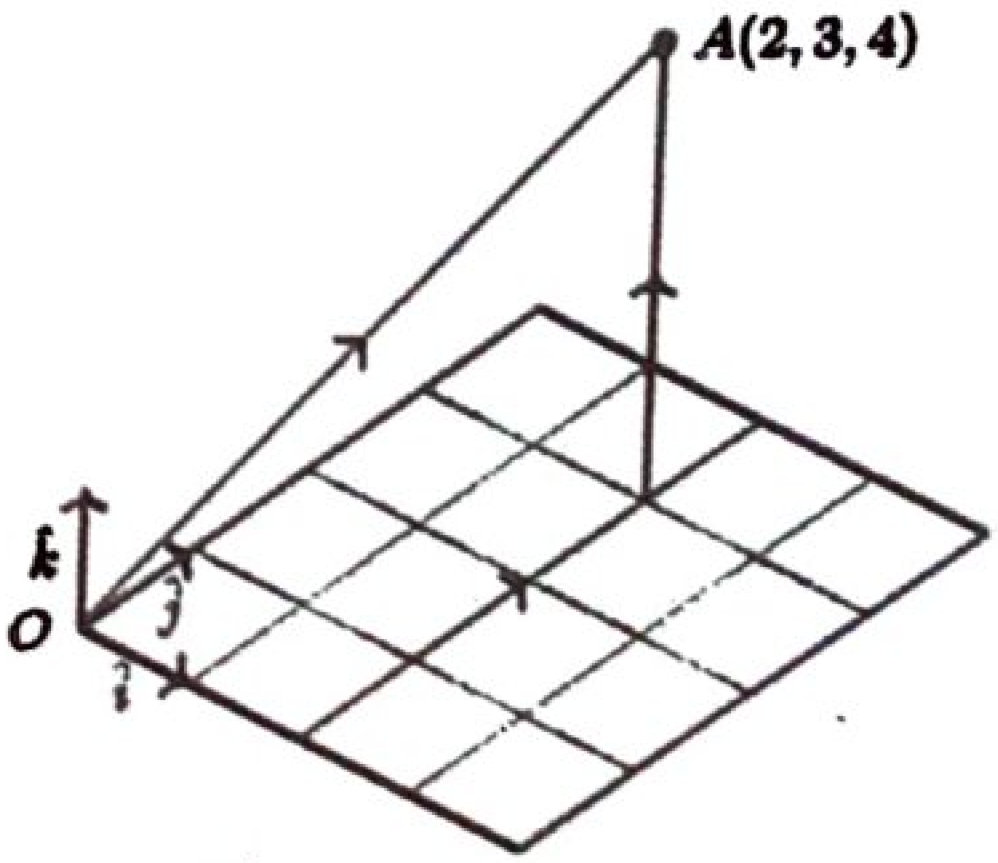
Position Vectors in Three Dimensions

If $\overrightarrow{OA}$ is a vector with initial point at the origin O and terminal point at point $A(2,3,4)$, then we can represent $\overrightarrow{OA}$ in terms of the vectors $\hat{i}, \hat{j}$ and $\hat{k}$ as

$$\overrightarrow{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}.$$

We can also represent vector in three dimensions using column vector as like in two dimensions.

$$\overrightarrow{OA} = \begin{pmatrix}2\\3\\4\end{pmatrix}$$

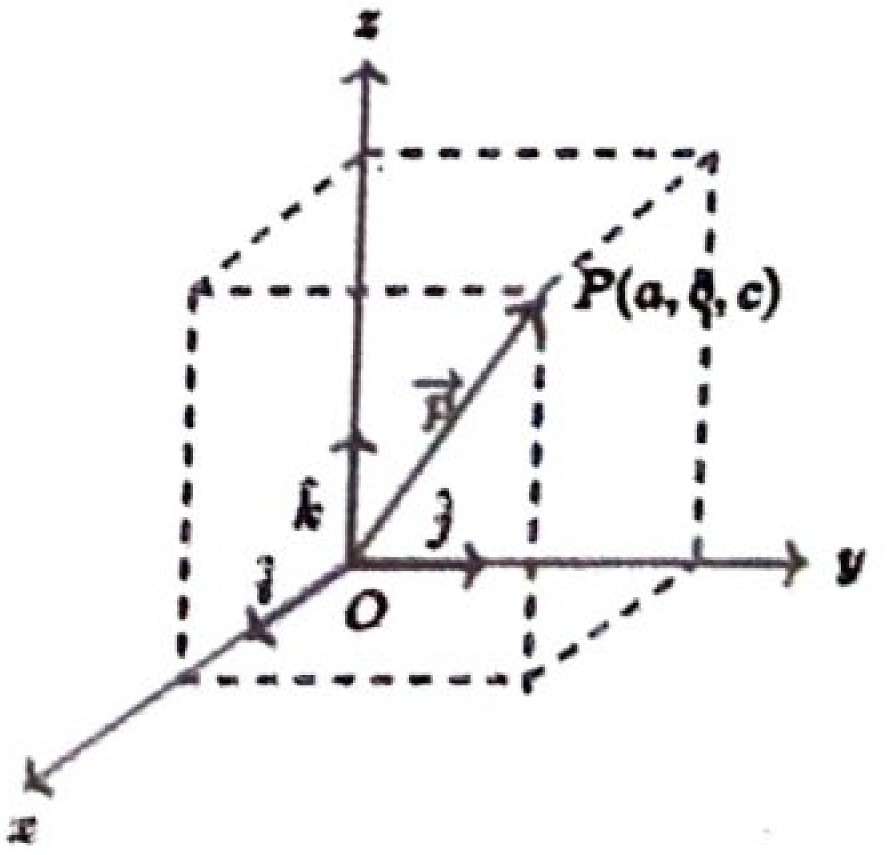


$\overrightarrow{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

The numbers in each column are called the **components** of the vector.

A vector whose initial point at the origin is called a **position vector**. For example, the position of point $P(a, b, c)$ in the diagram can be represented by its position vector, $\overrightarrow{OP}$. So,

$\vec{p} = \overrightarrow{OP} = \begin{pmatrix}a\\b\\c\end{pmatrix} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$ is the position vector of the point P.



The magnitude of a vector $\overrightarrow{OP} = \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is given by

$$\boxed{|\vec{r}| = \sqrt{a^2 + b^2 + c^2}.}$$

In general, for the position vectors of two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ find the vector $\overrightarrow{AB}$ as follows:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

and $\overrightarrow{AB}$ is called the position vector of B relative to A.

Example 2.

If P is $(-3, 1, 2)$ and Q is $(1, -1, 3)$, find:

(a) $\overrightarrow{OP}$ (b) $\overrightarrow{PQ}$ (c) $|\overrightarrow{PQ}|$ (d) $\overrightarrow{QP}$ (e) $|\overrightarrow{QP}|$

Solution

(a) $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

(b) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 3 \\ -1 - 1 \\ 3 - 2 \end{pmatrix} =$ [illegible]

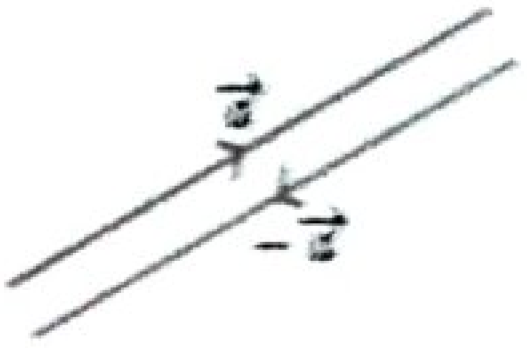
(c) $|\overrightarrow{PQ}| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}.$

(d) $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 - 1 \\ 1 + 1 \\ 2 - 3 \end{pmatrix} =$ [illegible]

(e) $|\overrightarrow{QP}| = \sqrt{(-4)^2 + 2^2 + (-1)^2} = \sqrt{21}.$

Algebraic Operations with Vectors

Negative Vector



In the diagram, the vector $-\vec{a}$ has the same length as the vector $\vec{a}$ but the opposite direction.

When $\vec{a}$ is given in component form, the components of $-\vec{a}$ are the same as those for $\vec{a}$ but with their sign reversed.

So
$$-\begin{pmatrix} 11 \\ -6 \\ 5 \end{pmatrix} = \begin{pmatrix} -11 \\ 6 \\ -5 \end{pmatrix}.$$

Zero Vector

In three dimensional space, **the zero vector** is denoted by $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The rule for algebra with vectors extend from two dimensions to three dimensions:

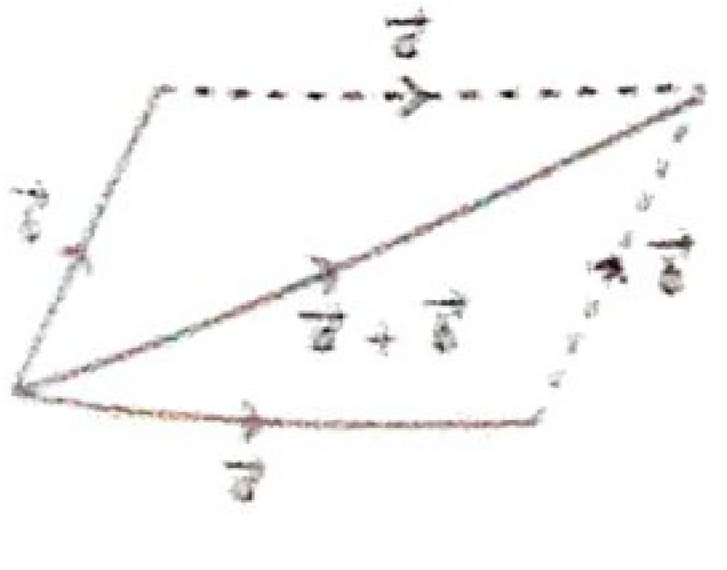
If $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, then

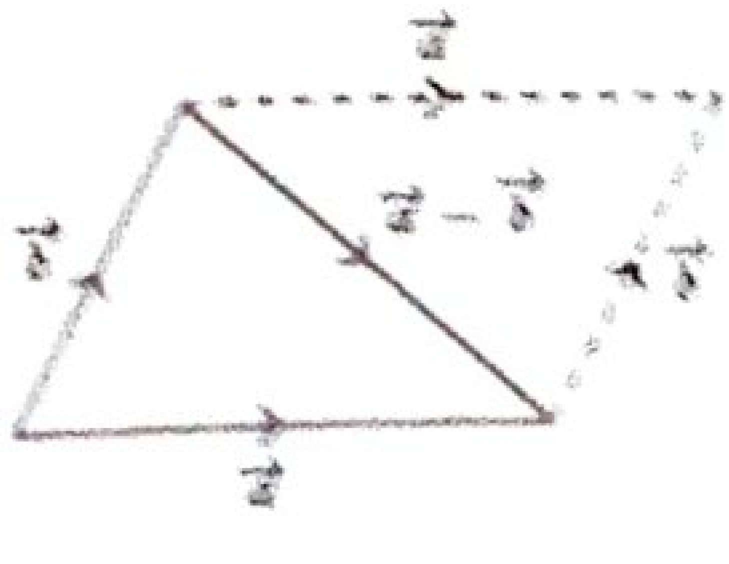
$$\vec{a} + \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix},$$

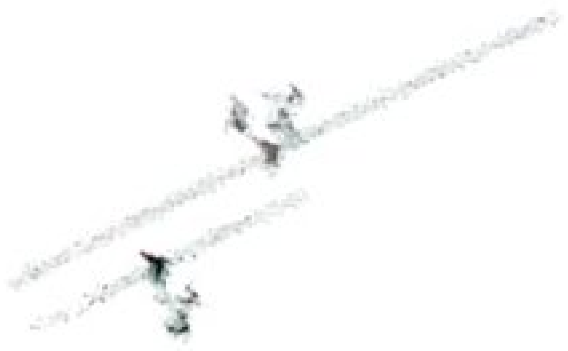
$$\vec{a} - \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix} \quad \text{and}$$

$$k\vec{a} = \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} \quad \text{for any scalar } k.$$

The following figures give the geometric interpretation of two vectors for addition subtraction and scalar multiplication of two vectors, respectively.







xample 3.

$\vec{p} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$, find:

) $\vec{p} + \vec{q}$ (b) $\vec{p} - \frac{1}{2}\vec{q}$ (c) $\frac{3}{2}\vec{q} - \vec{p}$.

olution

)
$$\vec{p} + \vec{q} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-2 \\ -1+0 \\ 4+2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}.$$

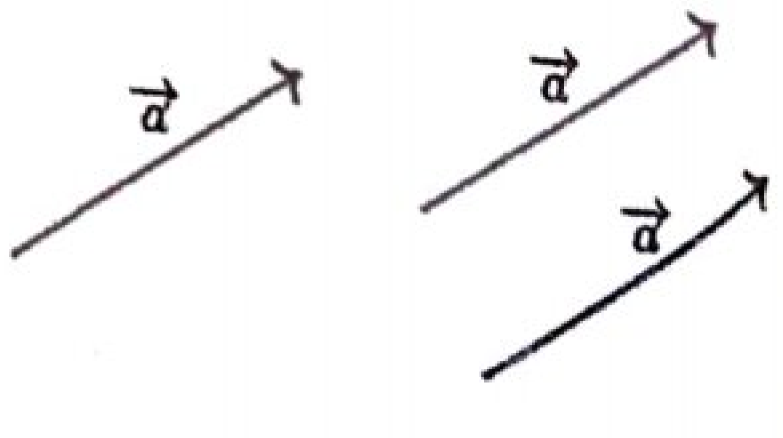
)
$$\vec{p} - \frac{1}{2}\vec{q} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+1 \\ -1-0 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

)
$$\frac{3}{2}\vec{q} - \vec{p} = \frac{3}{2}\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3-1 \\ 0+1 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}.$$

qual Vectors and Parallel Vectors

wo vectors are **equal** if they have the same magnitude and direction.

o, if arrows are used to represent vectors, then qual vectors are **parallel** and **equal in length.** his means that equal vector arrows are translations f one another, but in space.

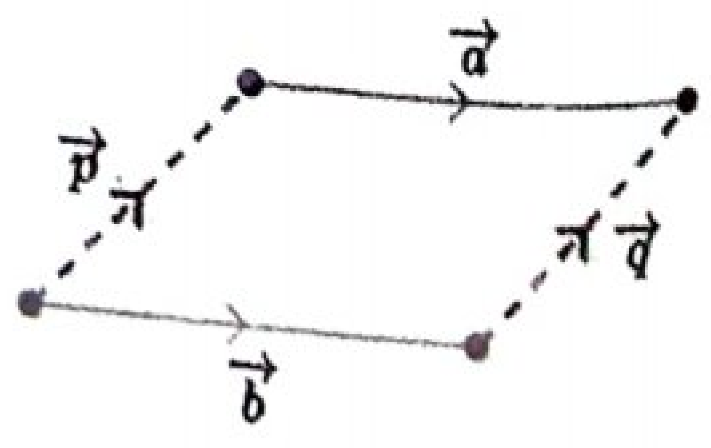


$\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, then $\vec{a} = \vec{b}$ if and only if $x_1 = x_2,\ y_1 = y_2,\ z_1 =$

 this diagram, we see that

$$\vec{a} = \vec{b},$$

$$\vec{p} = \vec{q}.$$



wo vectors $\vec{a}$ and $\vec{b}$ are **parallel** if and only if $\vec{a} = k\,\vec{b}$ for some scalar k.

Example 4.

Find u and v given that $\overrightarrow{a} = \begin{pmatrix} -1 \\ -1 \\ u \end{pmatrix}$ is parallel to $\overrightarrow{b} = \begin{pmatrix} v \\ 2 \\ -2 \end{pmatrix}$.

Solution

Since $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, we have $\overrightarrow{a} = k\,\overrightarrow{b}$ for some scalar k.

$$\begin{pmatrix} -1 \\ -1 \\ u \end{pmatrix} = k \begin{pmatrix} v \\ 2 \\ -2 \end{pmatrix}.$$

$$-1 = kv, \quad -1 = 2k \quad \text{and} \quad u = -2k.$$

We have $k = -\frac{1}{2}$.

Thus $u = 1$ and $v = 2$.

Example 5.

$ABCD$ is a parallelogram. A is $(-1, 1, 1)$, B is $(2, 0, -2)$, and D is $(3, 1, 4)$. Find the coordinates of C.

Solution

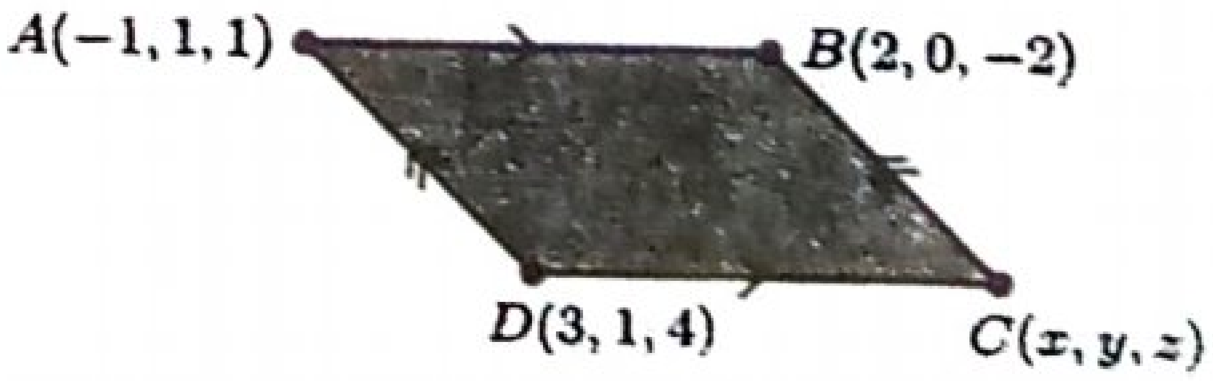
Let C be (x, y, z).

Since $ABCD$ is a parallelogram, $AB \parallel DC$, and they have the same length, so

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} x-3 \\ y-1 \\ z-4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$$

$$\begin{aligned} x - 3 &= 3, & y - 1 &= -1, & z - 4 &= -3 \\ x &= 6, & y &= 0, & z &= 1. \end{aligned}$$

Therefore C is $(6, 0, 1)$.

Unit Vector

A vector with magnitude 1 is called a **unit vector**.

For a nonzero vector $\overrightarrow{a}$, the unit vector in the (same) direction of $\overrightarrow{a}$, denoted by $\hat{a}$ is given by

$$\boxed{\hat{a} = \frac{\overrightarrow{a}}{|\overrightarrow{a}|}.}$$

Example 6.

(a) Find the unit vector in the same direction as $\overrightarrow{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 that is parallel to $\overrightarrow{a}$.

Solution

(a)
$$|\overrightarrow{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

The unit vector of $\overrightarrow{a}$ is

$$\hat{a} = \frac{\overrightarrow{a}}{|\overrightarrow{a}|} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

(b) Let $\overrightarrow{b}$ be parallel to $\overrightarrow{a}$ and $|\overrightarrow{b}| = 5$.

Then $\overrightarrow{b} = 5\hat{a} = \begin{pmatrix} \frac{10}{3} \\ -\frac{10}{3} \\ \frac{5}{3} \end{pmatrix}$ or $\overrightarrow{b} = -5\hat{a} = \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ -\frac{5}{3} \end{pmatrix}$.

Collinear Points

Three or more points are said to be **collinear** if they lie on the same straight line.
A, B and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

Example 7.

Prove that $A(8,2,2), C(20,5,5)$ and $B(12,3,3)$ are collinear.

Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 12 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 20 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 12 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$= 2\,\overrightarrow{AB}.$$

Therefore BC is parallel to AB.
Since B is common to both, it follows that A, B and C are collinear.

Exercise 4.1

1. Let $\overrightarrow{a} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$, $\overrightarrow{b} = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$ and $\overrightarrow{c} = \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix}$. Find the following vectors:
   - (a) $3\overrightarrow{a}$
   - (b) $4\overrightarrow{b}$
   - (c) $\overrightarrow{a} - \overrightarrow{b}$
   - (d) $\overrightarrow{b} + \overrightarrow{c}$
   - (e) $2\overrightarrow{b} + \overrightarrow{c}$
   - (f) $\overrightarrow{a} - 2\overrightarrow{b}$
   - (g) $\overrightarrow{a} + \overrightarrow{b} - 2\overrightarrow{c}$
   - (h) $3\overrightarrow{a} - \overrightarrow{b} + \overrightarrow{c}$

2. Given vectors $\overrightarrow{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\overrightarrow{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$.
   - (a) Find the values of p and q such that $\overrightarrow{c}$ is parallel to $\overrightarrow{a}$.
   - (b) Find the value of scalar k such that $\overrightarrow{a} + k\overrightarrow{b}$ is parallel to vector $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

3. Points A, B, C and D have position vectors $\vec{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 7 \\ 8 \\ -3 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$, respectively. Point E is the midpoint of BC.
   (a) Find the position vector of E.
   (b) Show that $ABED$ is a parallelogram.

4. Points A, B and C have position vectors $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, respectively. Find the position vector of point D such that $ABCD$ is a parallelogram.

5. $K(1, -1, 0), L(4, -3, 7)$ and $M(a, 2, b)$ are collinear. Find a and b.

4.2 Angle between Two Vectors and Scalar Product

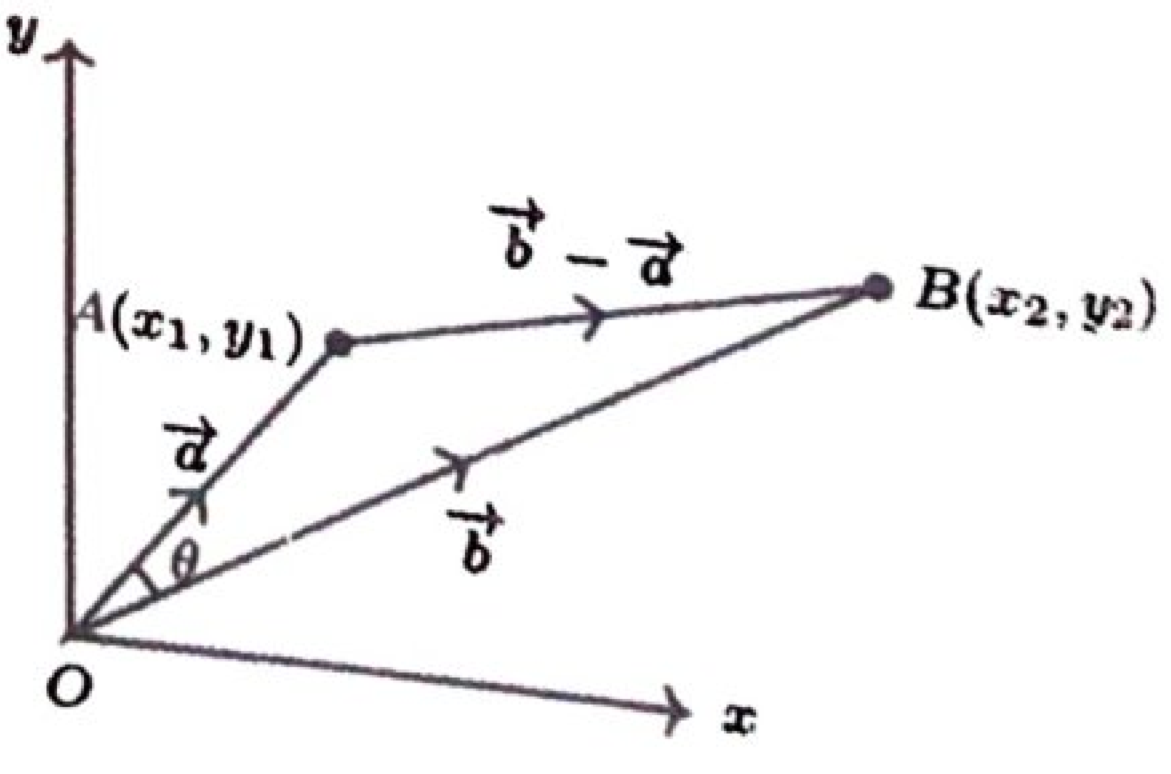
First, we consider the angle θ between the two vectors $\vec{a}$ and $\vec{b}$ for two dimensions.

Let $\overrightarrow{OA} = \vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \overrightarrow{OB} = \vec{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and θ be the angle between them.

In $\triangle OAB$, we use the cosine rule,

$$|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}|\,|\overrightarrow{OB}| \cos\theta,$$

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}| \cos\theta. \qquad (1)$$



But $\vec{b} - \vec{a} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$, equation (1) becomes

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2|\vec{a}|\,|\vec{b}| \cos\theta$$

which simplifies to

$$x_1x_2 + y_1y_2 = |\vec{a}|\,|\vec{b}| \cos\theta.$$

So, $$\cos\theta = \frac{x_1x_2 + y_1y_2}{|\overrightarrow{a}|\,|\overrightarrow{b}|}. \tag{2}$$

The expression $x_1x_2 + y_1y_2$ can be written by $\overrightarrow{a}\cdot\overrightarrow{b}$. That is,

$$\boxed{\overrightarrow{a}\cdot\overrightarrow{b} = x_1x_2 + y_1y_2}$$

for two dimensions. This method can be extended into three dimensions.

If $\overrightarrow{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, it can be written by
$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\overrightarrow{a}|\,|\overrightarrow{b}|}. \tag{3}$$

Thus $\overrightarrow{a}\cdot\overrightarrow{b}$ is also written by

$$\boxed{\overrightarrow{a}\cdot\overrightarrow{b} = x_1x_2 + y_1y_2 + z_1z_2.}$$

The expression $\overrightarrow{a}\cdot\overrightarrow{b}$ is called the **scalar product** or **dot product** of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

In both cases, we can write (2) and (3) as follows:

$$\boxed{\cos\theta = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|\,|\overrightarrow{b}|}.}$$

This result can be written in the form,

$$\boxed{\overrightarrow{a}\cdot\overrightarrow{b} = |\overrightarrow{a}|\,|\overrightarrow{b}|\cos\theta.}$$

Example 8.

Find the angle between the two vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

Solution

Let $\overrightarrow{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

We can find that $|\vec{a}| = \sqrt{3^2+4^2} = 5$ and $|\vec{b}| = \sqrt{5^2+(-12)^2} = 13.$

and $\vec{a}\cdot\vec{b} = \begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}5\\-12\end{pmatrix} = 3\times 5 + 4\times(-12) = 15 - 48 = -33.$

$$\cos\,\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\,\theta = \frac{-33}{65}$$

$$\theta = 120.5°.$$

Example 9.

Given points $P(1,0,-1), Q(2,4,1)$ and $R(3,5,6)$, find $\angle QPR$.

Solution

The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by θ in $\cos\,\theta = \dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}|\,|\overrightarrow{PR}|}$.

Since $\overrightarrow{PQ} = \begin{pmatrix}2\\4\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix}$,we get

$$|\overrightarrow{PQ}| = \sqrt{1^2+4^2+2^2} = \sqrt{21}.$$

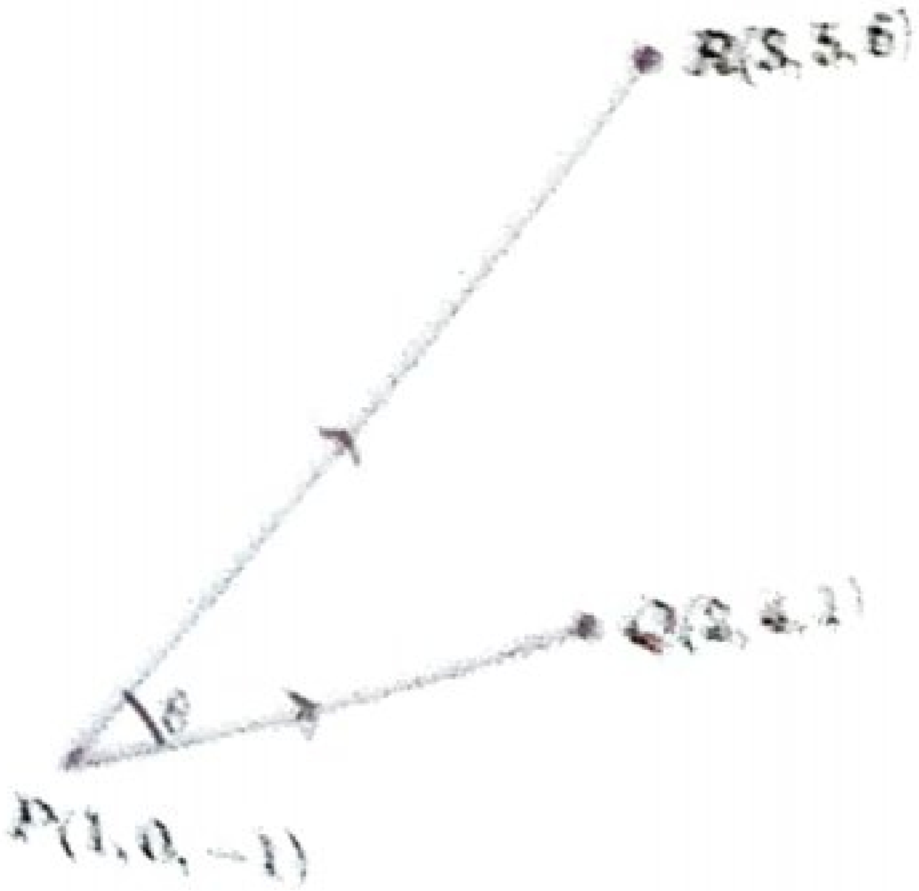
Again $\overrightarrow{PR} = \begin{pmatrix}3\\5\\6\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}2\\5\\7\end{pmatrix}$,we get

$$|\overrightarrow{PR}| = \sqrt{2^2+5^2+7^2} = \sqrt{78}.$$

$$\overrightarrow{PQ}\cdot\overrightarrow{PR} = \begin{pmatrix}1\\4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\5\\7\end{pmatrix} = 36,$$

$$\cos\,\theta = \frac{36}{\sqrt{21}\times\sqrt{78}},$$

$$\theta = 27.2°.$$



Algebraic Properties of the Scalar Product

The scalar product of two vectors has the following algebraic properties.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in space and k is a scalar, then

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,
2. $(-\vec{a}) \cdot \vec{b} = \vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b})$,
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$,
4. $(k\,\vec{a}) \cdot \vec{b} = k\,(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\,\vec{b})$,
5. $\vec{0} \cdot \vec{a} = 0$.

Geometric Properties of the Scalar Product

1. If nonzero vectors $\vec{a}$ and $\vec{b}$ are **perpendicular**, then $\vec{a} \cdot \vec{b} = 0$.
2. If nonzero vectors $\vec{a}$ and $\vec{b}$ are **parallel** in same direction, then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$, in particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
3. If nonzero vectors $\vec{a}$ and $\vec{b}$ are **parallel** in opposite directions, then $\vec{a} \cdot \vec{b} = -|\vec{a}|\,|\vec{b}|$.

Example 10.

Given that $\vec{a}$ and $\vec{b}$ are perpendicular vectors such that $|\vec{a}| = 3$ and $|\vec{b}| = 1$, evaluate $(\vec{a} - \vec{b}) \cdot (\vec{a} + 5\vec{b})$.

Solution

Since $\vec{a}$ and $\vec{b}$ are perpendicular, so $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 0$.

$$\begin{aligned}(\vec{a} - \vec{b}) \cdot (\vec{a} + 5\vec{b}) &= \vec{a} \cdot \vec{a} + 5\,(\vec{a} \cdot \vec{b}) - \vec{b} \cdot \vec{a} - 5\,(\vec{b} \cdot \vec{b}) \\ &= \vec{a} \cdot \vec{a} - 5\,(\vec{b} \cdot \vec{b}) \\ &= |\vec{a}|^2 - 5|\vec{b}|^2 = 3^2 - 5 \times 1^2 = 4.\end{aligned}$$

Example 11.

Points A, B and C have position vectors $\vec{a} = k\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}3\\2\\-2\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}1\\1\\4\end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$.

Solution

(a) $\overrightarrow{BC} = \vec{c} - \vec{b} = \begin{pmatrix}1\\1\\4\end{pmatrix} - \begin{pmatrix}3\\2\\-2\end{pmatrix} = \begin{pmatrix}-2\\-1\\6\end{pmatrix}$.

(b) $\overrightarrow{AB} = \vec{b} - \vec{a} = \begin{pmatrix}3\\2\\-2\end{pmatrix} - k\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}3-2k\\2+k\\-2-k\end{pmatrix}$.

(c) Since $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$, it follows that

$$\begin{aligned}
\overrightarrow{AB}\cdot\overrightarrow{BC} &= 0\\
\begin{pmatrix}3-2k\\2+k\\-2-k\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\6\end{pmatrix} &= 0\\
-6+4k-2-k-12-6k &= 0\\
-3k &= 20\\
k &= -\frac{20}{3}.
\end{aligned}$$

Exercise 4.2

1. For $\vec{p} = \begin{pmatrix}3\\2\end{pmatrix}$, $\vec{q} = \begin{pmatrix}-1\\5\end{pmatrix}$ and $\vec{r} = \begin{pmatrix}-2\\4\end{pmatrix}$, find:

   (a) $\vec{q}\cdot\vec{p}$

   (b) $\vec{q}\cdot\vec{r}$

   (c) $\vec{q}\cdot(\vec{p}+\vec{r})$

(d) $\hat{i} \cdot \vec{p}$ (e) $\vec{q} \cdot \hat{j}$ (f) $\hat{i} \cdot \hat{i}$.

2. For $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, find:

(a) $\vec{a} \cdot \vec{b}$ (b) $\vec{b} \cdot \vec{a}$ (c) $|\vec{a}|^2$

(d) $\vec{a} \cdot \vec{a}$ (e) $\vec{a} \cdot (\vec{b} + \vec{c})$ (f) $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

3. Find the angle between $\vec{m}$ and $\vec{n}$ if:

(a) $\vec{m} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and $\vec{n} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ (b) $\vec{m} = 2\hat{j} - \hat{k}$ and $\vec{n} = \hat{i} + 2\hat{k}$.

4. Find t if the given pair of vectors are:

(i) perpendicular (ii) parallel.

(a) $\vec{p} = \begin{pmatrix} 3 \\ t \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, (b) $\vec{r} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$ and $\vec{s} = \begin{pmatrix} t \\ -4 \end{pmatrix}$,

(c) $\vec{a} = \begin{pmatrix} 0 \\ t+2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2-3t \\ t \end{pmatrix}$.

5. Find t if $\begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix}$

6. $ABCD$ is a parallelogram with AB parallel to DC. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$.

(a) Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\vec{a}$ and $\vec{b}$

(b) Simplify $(\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a})$

(c) Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular.

4.3 Area of a Parallelogram and Vector Product

The vector product is only defined when both vectors are three-dimensional. The **vector product (or cross product)** of $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, is given by

$$\vec{a} \times \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}.$$

Instead of memorizing the formula for the vector product try using the following shortcut.

(1) Eliminate the component column that you are trying to calculate.

(2) Calculate: Down Product − Up Product.

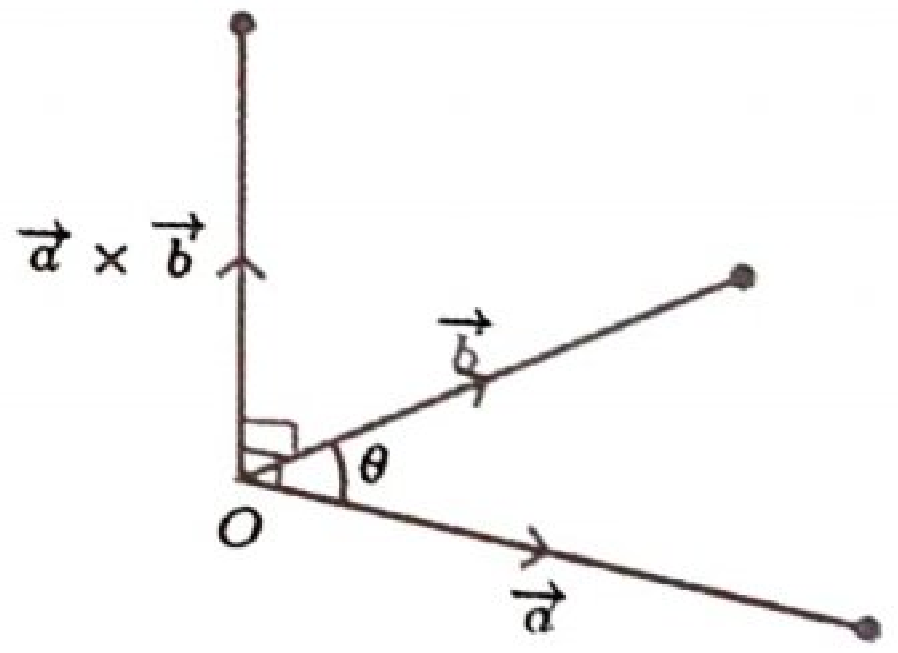
$$\vec{a}\times\vec{b} = \left\{ \begin{array}{ccccccccc} x_1 & & y_1 & & z_1 & & x_1 & & y_1 \\ & & & \times & & \times & & \times & \\ x_2 & & y_2 & & z_2 & & x_2 & & y_2 \end{array} \right.$$

The direction of $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ as shown in the diagram.

We have known that the nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}\cdot\vec{b}=0$.

We will examine: $(\vec{a}\times\vec{b})\cdot\vec{a}$.

Let $\vec{a}=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b}=\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.



Then $(\vec{a}\times\vec{b})\cdot\vec{a}=\begin{pmatrix} y_1z_2-z_1y_2 \\ z_1x_2-x_1z_2 \\ x_1y_2-y_1x_2 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$$\begin{aligned} &=(y_1z_2-z_1y_2)\,x_1+(z_1x_2-x_1z_2).y_1+(x_1y_2-y_1x_2)z_1 \\ &=x_1y_1z_2-x_1y_2z_1+x_2y_1z_1-x_1y_1z_2+x_1y_2z_1-x_2y_1z_1 \\ &=0. \end{aligned}$$

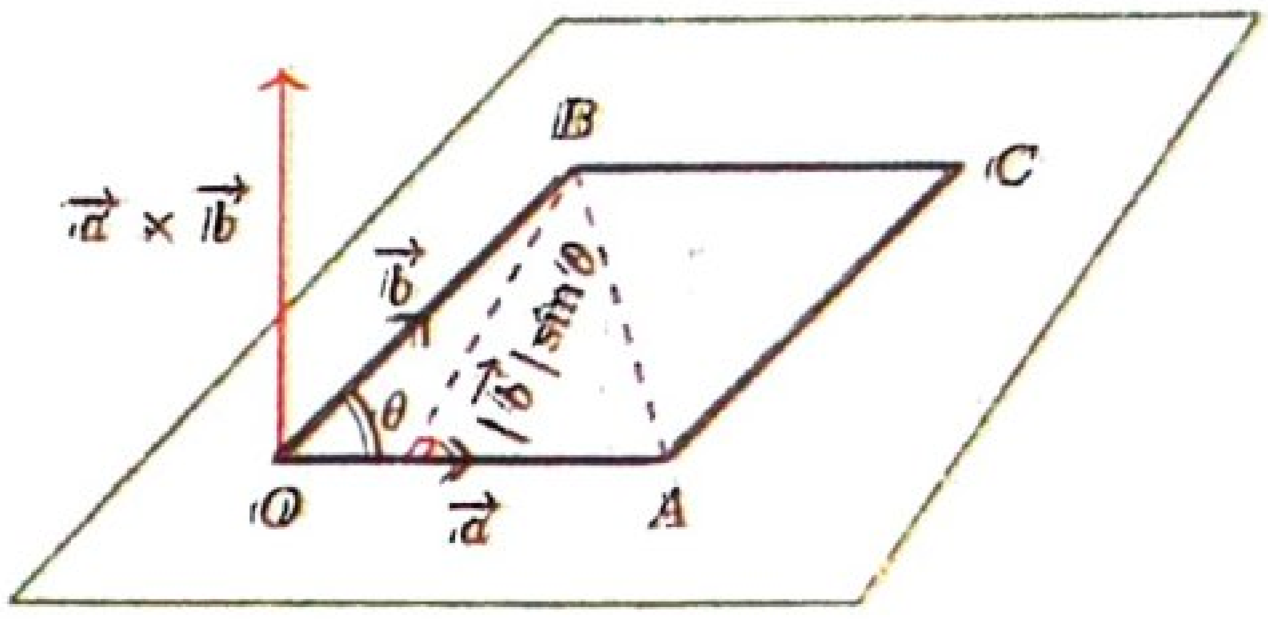
Therefore $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$. Similarly, we can show that $\vec{a}\times\vec{b}$ is also perpendicular to $\vec{b}$.

The vector product of $\vec{a}=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b}=\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, has magnitude $|\vec{a}|\,|\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Area of a Parallelogram

Consider the parallelogram $OACB$ lying on the plane, with the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ as shown in the diagram. Then, the area of $OACB$ is given by

$$OA\ |\overrightarrow{b}|\sin\theta = |\overrightarrow{a}|\ (|\overrightarrow{b}|\ \sin\ \theta) = |\overrightarrow{a}\times\overrightarrow{b}|,$$


$\overrightarrow{a}\times\overrightarrow{b}$
B
C
$\overrightarrow{b}$
$|\overrightarrow{b}|\sin\theta$
θ
O
$\overrightarrow{a}$
A

i.e., the area of the parallelogram with sides defined by vectors $\overrightarrow{OA} = \overrightarrow{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overrightarrow{OB} = \overrightarrow{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ is equal to the magnitude of the vector $\overrightarrow{a}\times\overrightarrow{b}$.

Consequently,

$$\alpha(\triangle ACB) = \frac{1}{2}\,(\alpha(OACB)) = \frac{1}{2}\,|\overrightarrow{a}\times\overrightarrow{b}|.$$

Example 12.

Find the area of the parallelogram determined by the vectors $\overrightarrow{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

Solution

We can find that $\overrightarrow{a}\times\overrightarrow{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2\times(-1) - 3\times 4 \\ 3\ \times\ 1\ \ - 1\times(-1) \\ 1\ \times\ 4\ \ - 2\times 1 \end{pmatrix} = \begin{pmatrix} -14 \\ 4 \\ 2 \end{pmatrix}.$$

Now, $|\overrightarrow{a}\times\overrightarrow{b}| = \sqrt{196+16+4} = \sqrt{216}.$

Therefore the area of the parallelogram is $\sqrt{216}$ unit^2.

Example 13.

Find the area of the triangle ABC with vertices $A(1,-1,3)$, $B(0,4,1)$ and $C(2,7,2$

Solution

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}0\\4\\1\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}-1\\5\\-2\end{pmatrix} \text{ and}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}2\\7\\2\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}1\\8\\-1\end{pmatrix}.$$

Then, we find their vector product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-1\\5\\-2\end{pmatrix}\times\begin{pmatrix}1\\8\\-1\end{pmatrix}=\begin{pmatrix}11\\-3\\-13\end{pmatrix}.$$

Next $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{11^2+(-3)^2+(-13)^2}=\sqrt{299}.$

Therefore, we have

Area of triangle $ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{299}$ unit^2.

Algebraic Properties of the Vector Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in space and k is a scalar, then

1. $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$,
2. $(k\vec{a})\times\vec{b}=k(\vec{a}\times\vec{b})$,
3. $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$,
4. $\vec{a}\times\vec{0}=\vec{0}$.

eometric Properties of the Vector Product

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1. If two nonzero vectors $\vec{a}$ and $\vec{b}$ are **parallel**, then $\vec{a} \times \vec{b} = \vec{0}$.
   In particular, $\vec{a} \times \vec{a} = \vec{0}$.
2. If two nonzero vectors $\vec{a}$ and $\vec{b}$ are **perpendicular**, then $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}$

---

xample 14.

a) Calculate $\vec{a} \times \vec{b}$ when $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} + 2\hat{k}$.

b) Find a unit vector $\hat{n}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$.

olution

a) $\vec{a} \times \vec{b} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 2 - 5 \times (-4) \\ 5 \times 1 - 3 \times 2 \\ 3 \times (-4) - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$.

b) $\vec{a} \times \vec{b} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$. This vector is perpendicular to $\vec{a}$ and $\vec{b}$.

$$|\vec{a} \times \vec{b}| = \sqrt{24^2 + (-1)^2 + (-14)^2} = \sqrt{773}.$$

So, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\hat{n} = \dfrac{1}{\sqrt{773}} \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$.

Example 15.

Given that $|\vec{a}| = 4, |\vec{b}| = 5$ and that $\vec{a}$ and $\vec{b}$ are perpendicular, evaluate $|(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b})|$.

Solution

$$\begin{aligned}(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b}) &= 2\vec{a} \times \vec{a} + 2\vec{a} \times 3\vec{b} - \vec{b} \times \vec{a} - \vec{b} \times 3\vec{b} \\ &= 2(\vec{a} \times \vec{a}) + 6(\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - 3(\vec{b} \times \vec{b}) \\ &= 6(\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a})\end{aligned}$$

$$
\begin{aligned}
&= 6(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \\
&= 7(\vec{a} \times \vec{b}). \\
|(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b})| &= 7|\vec{a} \times \vec{b}| \\
&= 7|\vec{a}|\,|\vec{b}| \\
&= 140.
\end{aligned}
$$

Exercise 4.3

1. Find a vector perpendicular to the following pair of vectors:

   (a) $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

2. Consider $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

   (a) Find $\vec{a} \times \vec{b}$.

   (b) Find $\sin\theta$ using $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$.

3. Prove that for any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2$

4. Given points A, B and C with coordinates $(3, -5, 1), (7, 7, 2)$ and $(-1, 1, 3)$

   (a) Calculate $\vec{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\vec{q} = \overrightarrow{BA} \times \overrightarrow{BC}$.

   (b) What can you say about vectors $\vec{p}$ and $\vec{q}$?

5. The points $A(3, 1, 2), B(-1, 1, 5)$ and $C(7, 2, 3)$ are vertices of a parallelog $ABCD$.

   (a) Find the coordinates of D.

   (b) Calculate the area of the parallelogram.

4.4 Lines and Planes in Space

Lines in Three-Dimensional Space

A **direction vector** of a straight line is a vector parallel to the line.

In three dimensions geometry we can determined the **equation of a line using** its **direction** and any **fixed point** on the line.

Suppose a line passes through a fixed point A with $\overrightarrow{OA} = \vec{a}$, and that the line is parallel to a vector $\vec{b}$.

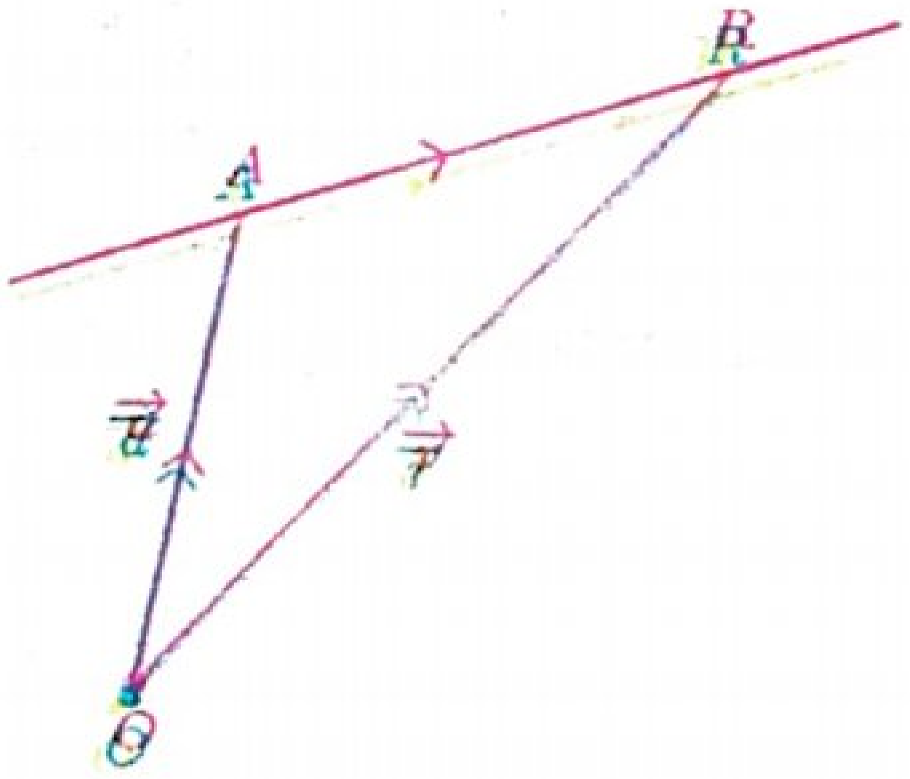
Let R be any point on the line with $\overrightarrow{OR} = \vec{r}$.

Then, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$.

Since $\overrightarrow{AR} \parallel \vec{b}$,

$$\overrightarrow{AR} = t\vec{b} \text{ for some } t \in \mathbb{R},$$

$$\vec{r} = \vec{a} + t\,\vec{b}.$$



So, $\vec{r} = \vec{a} + t\vec{b},\ t \in \mathbb{R}$ is the **vector equation** of the line.

In **three dimensions,** let $\overrightarrow{OR} = \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **vector equation** of the line where $R(x, y, z)$ is any point on the line, $A(a_1, a_2, a_3)$ is the known (fixed) point on the line and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **direction vector** of the line.

- Since $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix}$, we can write the **parametric equation** of the line as $\begin{cases} x = a_1 + tb_1 \\ y = a_2 + tb_2 \\ z = a_3 + tb_3 \end{cases}$ where $t \in \mathbb{R}$ is the **parameter.**

Each point on the line corresponds to exactly one value of t.

- By equating t values, we obtain the **Cartesian equation** of the line

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}.$$

xample 16.

ind the Cartesian equation of the line with vector equation $\vec{r} = \begin{pmatrix}1\\4\\-1\end{pmatrix} + t$

olution

he vector equation is $\vec{r} = \vec{a} + t\,\vec{b}$, therefore

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\4\\-1\end{pmatrix} + t\begin{pmatrix}3\\2\\5\end{pmatrix}, t \in \mathbf{R}.$$

he parametric equations are:

$$\begin{aligned} x &= 1+3t, & t &= \frac{x-1}{3}, \\ y &= 4+2t, & t &= \frac{y-4}{2}, \\ z &= -1+5t, & t &= \frac{z+1}{5}. \end{aligned}$$

he Cartesian equation of the line is $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z+1}{5}$.

xample 17.

oes the point $A(3,-2,2)$ lie on the line with equation $\frac{x+1}{2} = \frac{4-y}{3} = \frac{2z}{3}$

olution

onsider the given Cartesian equation passing through the point $A(3,-2,2)$.

$$\frac{x+1}{2} = \frac{3+1}{2} = 2$$

$$\frac{4-y}{3} = \frac{4+2}{3} = 2$$

$$\frac{2z}{3} = \frac{2\times 2}{3} = \frac{4}{3}$$

id so

$$2 = 2 \neq \frac{4}{3}.$$

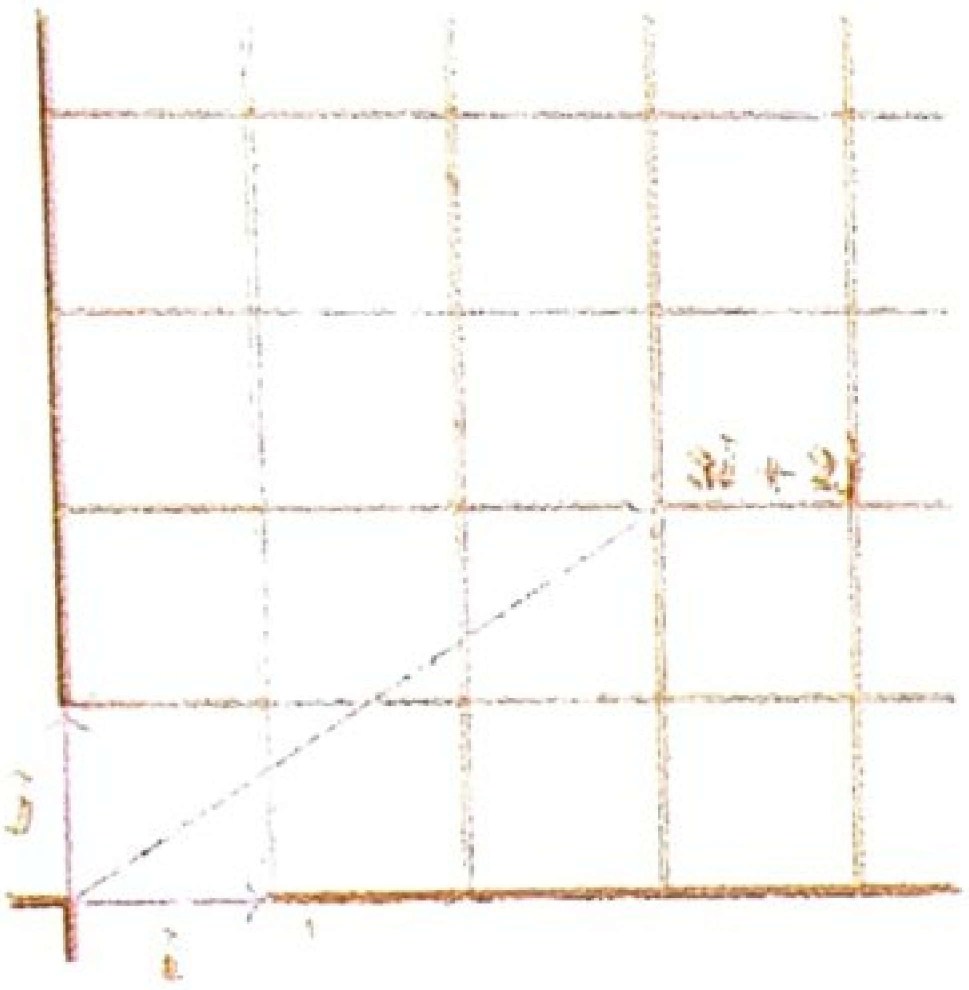
he coordinates do not satisfy the Cartesian

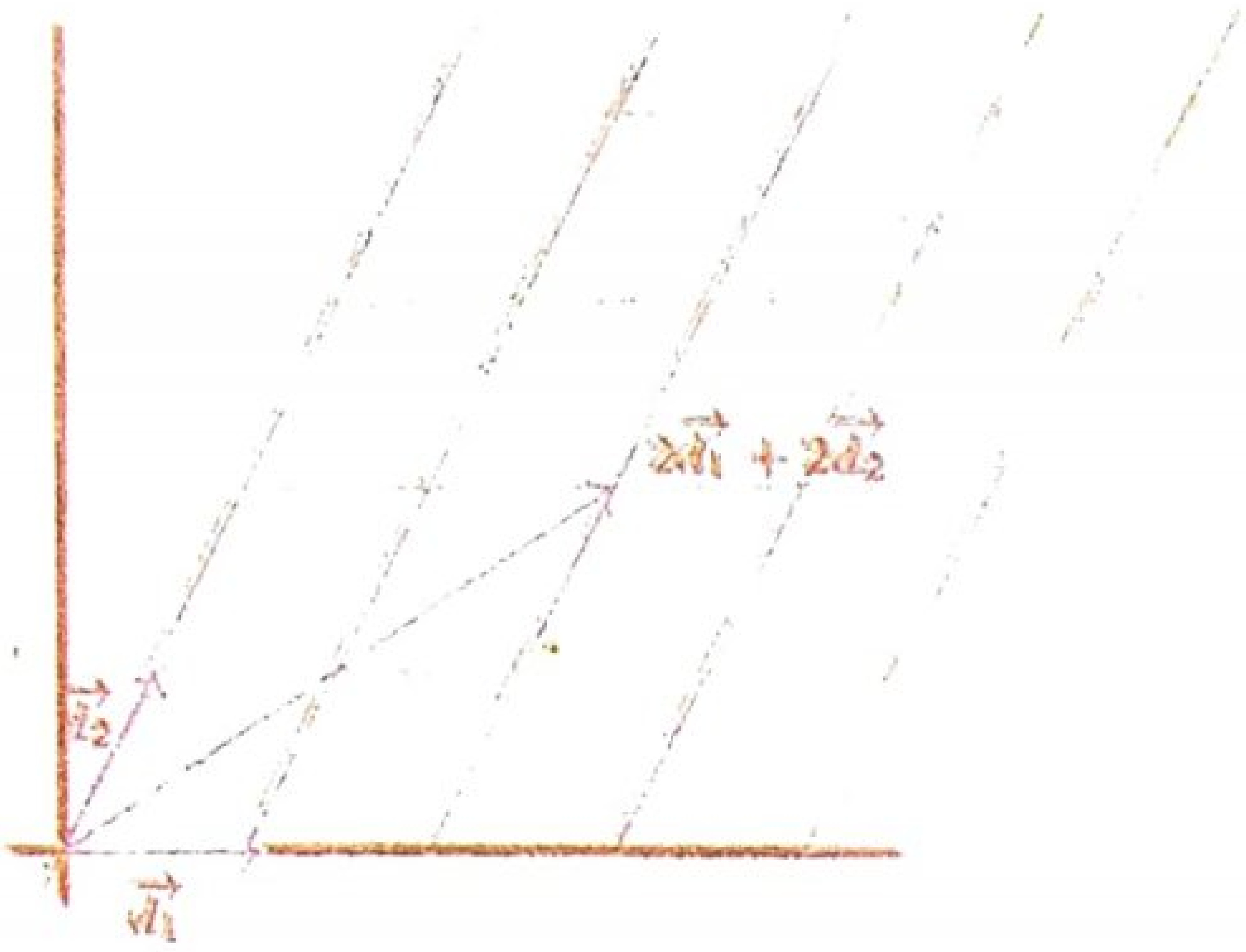
on the line.

Planes in Three Dimensions

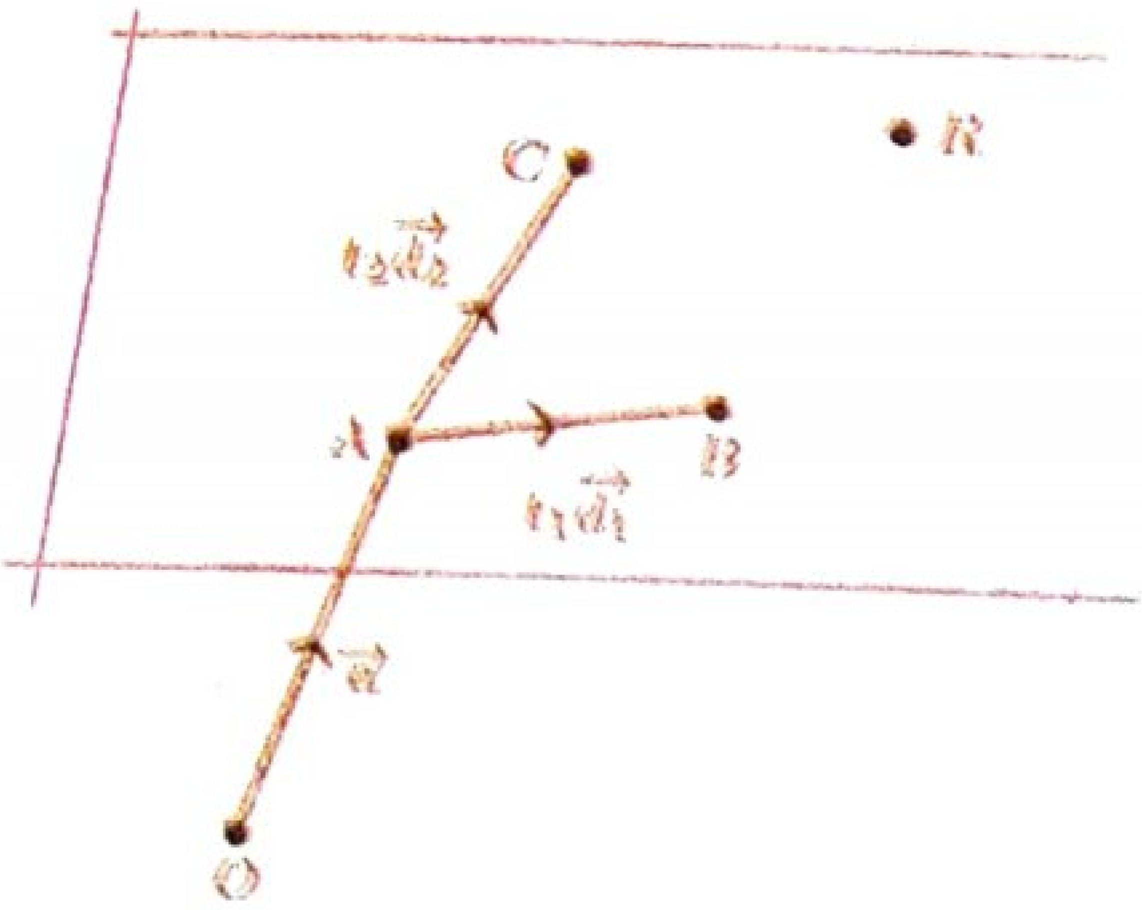
To determine the vector equation of the plane, we requires extension of the ideas of the equation of line. Think of a very simple example, the xy-plane. The position vector of any point in the xy-plane is a sum of scalar multiples of $\hat{i}$ and $\hat{j}$, so $\hat{i}$ and $\hat{j}$ can be considered direction vectors for the xy-plane.

More generally, for any plane through the origin, if we fix two nonparallel vectors in that plane, the position vector of any point in the plane is a sum of scalar multiples of those two vectors.





If the plane does not pass through the origin, we have to make a further turn: we need to shift the plane and its vectors by adding the position vector of some point on the plane.



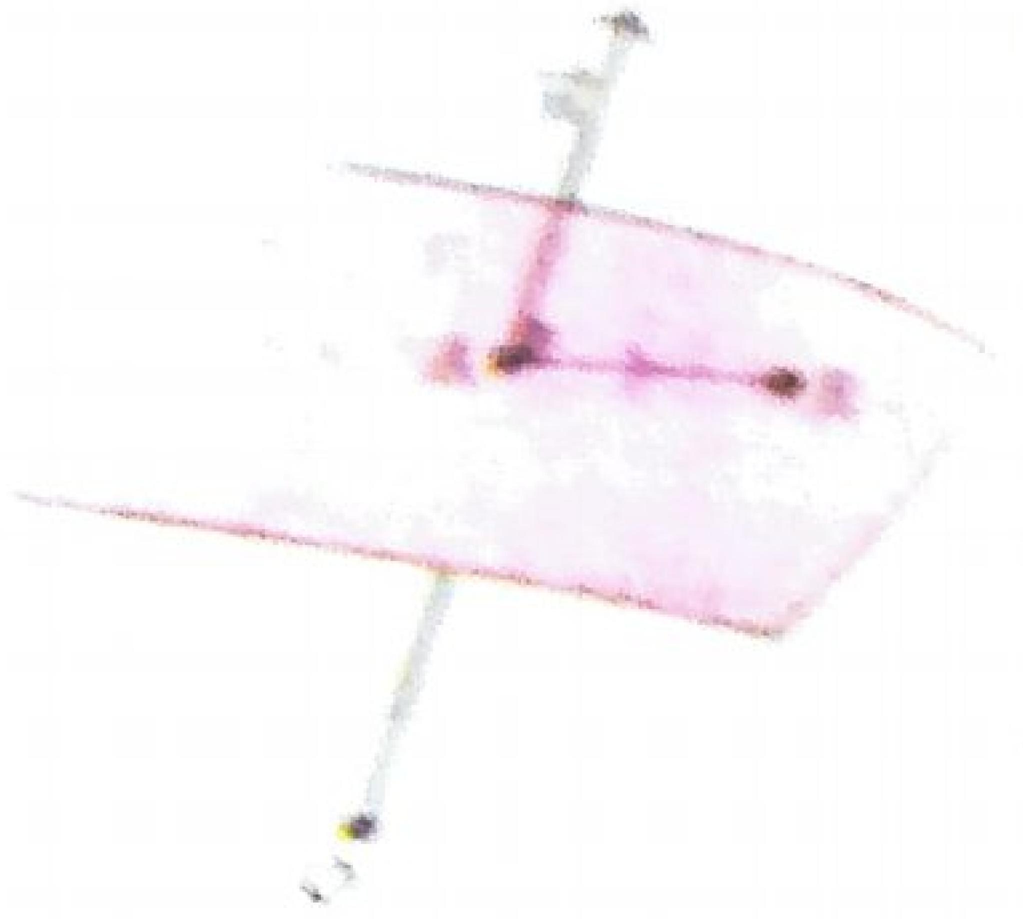
Consider A, B and C be three noncollinear points on the plane with $\overrightarrow{OA} = \vec{a}$. Let $\vec{d_1}$ and $\vec{d_2}$ be the two nonparallel vectors on the plane.

If the plane passing through A and having two nonparallel vectors $\vec{d_1}$ and $\vec{d_2}$, then the position vector $\vec{r}$ of any point R on the plane is given by following equation.

$$\boxed{\vec{r} = \vec{a} + t_1\vec{d_1} + t_2\vec{d_2}}$$

Any vector that is perpendicular to a plane is called a **normal vector** or simply **normal** to the plane. We can find the normal to a plane by finding the cross product of the two nonparallel vectors of the plane. Normal is perpendicular to every line on the plane.

To find the different vector form of the equation of a plane, we can use the po vector $\overrightarrow{a}$ of one point and the normal vector $\overrightarrow{n}$ is perpendicular to the plane.



Thus, we consider a plane passing through a point A with position vector $\overrightarrow{a}$ and $\overrightarrow{n} = a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular to the given plane. Let $\overrightarrow{r}$ be the position vector of an arbitrary point $R(x, y, z)$ on the plane. Since AR is a line in the plane, it follows that AR is at right angle to the normal vector $\overrightarrow{n}$.

$$\overrightarrow{n} \cdot \overrightarrow{AR} = 0.$$

The vector $\overrightarrow{AR}$ is given by $\overrightarrow{AR} = \overrightarrow{r} - \overrightarrow{a}$ and so

$$(\overrightarrow{r} - \overrightarrow{a}) \cdot \overrightarrow{n} = 0.$$

This is the vector equation of the plane. We can write this in an alterna form as

$$\overrightarrow{r} \cdot \overrightarrow{n} - \overrightarrow{a} \cdot \overrightarrow{n} = 0$$

$$\overrightarrow{r} \cdot \overrightarrow{n} = \overrightarrow{a} \cdot \overrightarrow{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \overrightarrow{a} \cdot \overrightarrow{n}$$

$$ax + by + cz = d \quad \text{where} \quad d = \overrightarrow{a} \cdot \overrightarrow{n}.$$ Note that d is a constant.

The Cartesian **equation** of a plane has the form $ax + by + cz = d$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector of the plane.

Example 18.

Find the vector equation of the plane containing points $M(2, 2, -2)$, $N(1,$ $P(4, 0, 2)$.

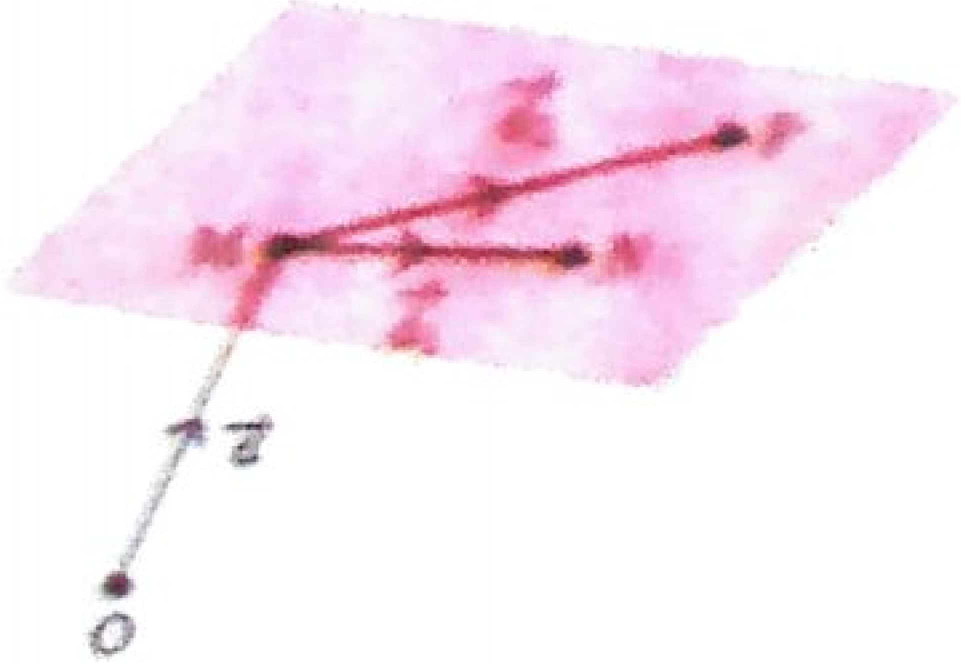
Solution

Let $\overrightarrow{a} = \overrightarrow{OM} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$. We can find that

$$\vec{d_1} = \overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix},$$

$$\vec{d_2} = \overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}.$$

Therefore the vector equation of the plane is



$$\vec{r} = \vec{a} + t_1\vec{d_1} + t_2\vec{d_2}$$

$$= \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}.$$

Example 19.

Find the Cartesian equation of the plane containing points $M(2, 2, -2)$, $N(1, -1, 3)$ and $P(4, 0, 2)$.

Solution

The equation of the plane containing M, N and P is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}. \text{ (See Example 18)}$$

$$x = 2 - t_1 + 2t_2 \quad (1)$$

$$y = 2 - 3t_1 - 2t_2 \quad (2)$$

$$z = -2 + 5t_1 + 4t_2 \quad (3)$$

Add equations (1) and (2),

$$x + y = 4 - 4t_1 \quad (4)$$

Multiply equation (2) by 2 and then add (3),

$$2y + z = 2 - t_1 \quad (5)$$

From equations (4) and (5), we obtain

$$\frac{x+y-4}{-4}=\frac{2y+z-2}{-1}$$

$$-x-y+4=-8y-4z+8$$

$$x-7y-4z=-4.$$

Example 20.

Determine whether points $A(3,-1,4), B(2,1,1), C(4,3,1)$ and $D(-3,1,4)$ the same plane.

Solution

The equation of the plane containing A, B and C is

$$\vec{r}=\overrightarrow{OA}+t_1\overrightarrow{AB}+t_2\overrightarrow{AC}$$

$$=\begin{pmatrix}3\\-1\\4\end{pmatrix}+t_1\begin{pmatrix}-1\\2\\-3\end{pmatrix}+t_2\begin{pmatrix}1\\4\\-3\end{pmatrix}.$$

If D is on the plane, then $\vec{r}=\overrightarrow{OD}$,

$$3-t_1+t_2 = -3$$

$$-1+2t_1+4t_2 = 1$$

$$4-3t_1-3t_2 = 4$$

Multiply equation (1) by 2, we get

$$-2t_1+2t_2=-12.$$

From (2) and (4), we obtain

$$t_1=\frac{13}{3},\ t_2=-\frac{5}{3}.$$

Substitute the values of t_1 and t_2 in left hand side of equation (3), we get

$$4-3(\frac{13}{3})-3(-\frac{5}{3})=-4\neq 4.$$

So, D does not lie on the same plane as A, B and C.

Example 21.

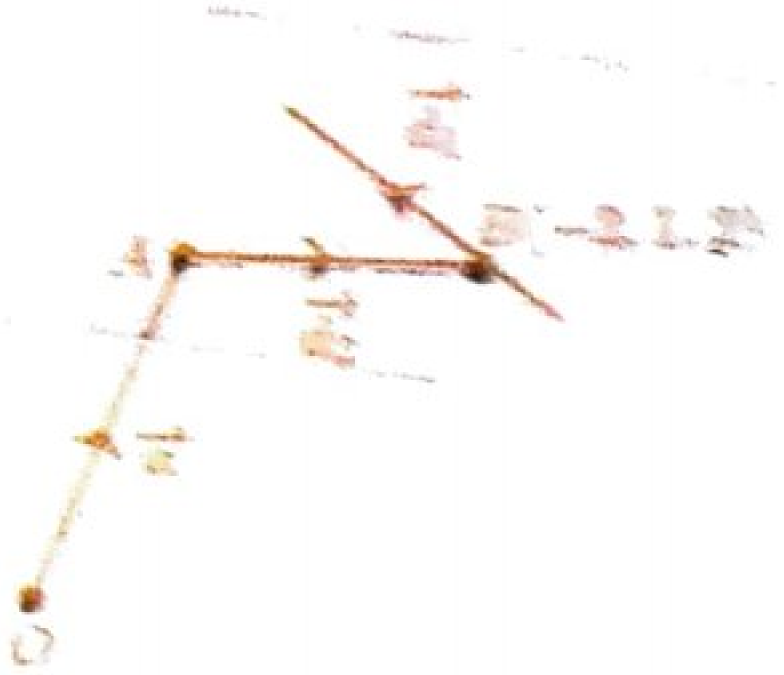
Find a vector equation of the plane containing the line $\vec{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ a

point $A(3, -1, 2)$.

Solution

Let $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

The vector $\vec{d_1} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is on the plane.



$B(-2, 1, 2)$ is on the plane and so

$$\vec{d_2} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}.$$

We see that $\vec{d_2}$ is also on the plane. Therefore the vector equation of the plane is

$$\vec{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}.$$

Example 22.

Vector $\vec{n} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$ is perpendicular to the plane which contains point $A(1, -5, 2)$.

(a) Write an equation of the plane in the form $\vec{r} \cdot \vec{n} = d$.

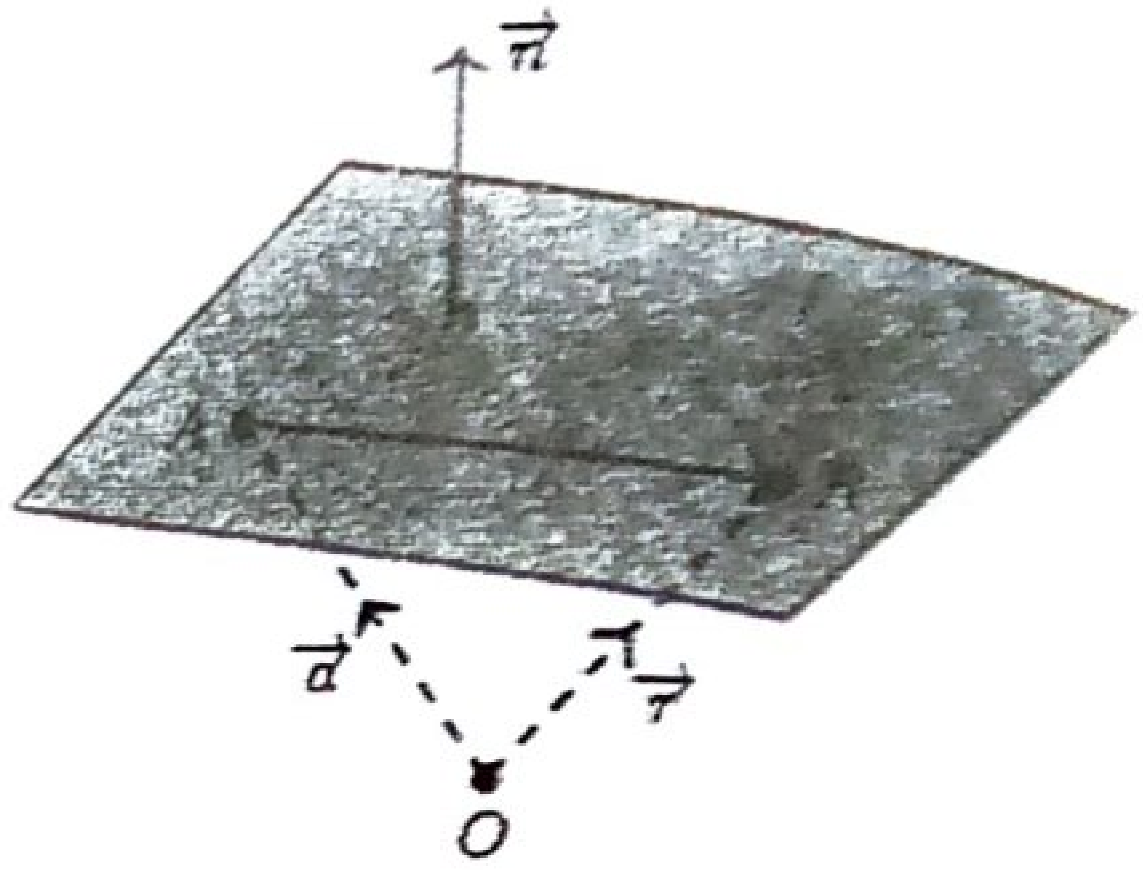
(b) Find the Cartesian equation of the plane.

Solution

(a) The vector equation of the plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$.

$$\vec{r} \cdot \vec{n} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = 2 - 20 - 4 = -22$$

(b)
$$\vec{r}\cdot\vec{n} = d$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = -22$$
$$2x + 4y - 2z = -22$$



Exercise 4.4

1. Find the vector equation of the line:
   (a) parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and through the point$(1, 3, -7)$.
   (b) through $(0, 1, 2)$ and with direction vector $\hat{i} + \hat{j} - 2\hat{k}$.
   (c) parallel to the x-axis and through the point $(-2, 2, 2)$.
2. (a) Find the Cartesian equation of the line with parametric equation $x = 3t + 1,\ y = 4 - 2t,\ z = 3t - 1$.
   (b) Find the unit vector in the direction of the line.
3. Find the equation of the plane:
   (a) with normal vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and which passes through $(-1, 2, 4)$.
   (b) perpendicular to the line joining points $A(2, 3, 1)$ and $B(5, 7, 2)$ and which passes through A.
   (c) containing $A(3, 2, 1)$ and the line $x = 1 + t,\ y = 2 - t,\ z = 3 + 2t$.
4. Find the equation of the plane through $A(-1, 2, 1)$, $B(4, 1, 1)$ and $C(2, 0, 3)$:
   (a) in vector form (b) in Cartesian form.
5. Find the Cartesian equation of the plane with vector equation
$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$

Chapter 8

Permutations and Combinations

[illegible]

8.1 Counting Principles

[illegible]

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	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
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In this problem, there are 2 ways to choose the type of drink. For each such choice, there are 3 ways to choose the type of snack. The number of ways to have a drink and a snack is 6, which can be computed as $2 \times 3 = 6$.

Now, let us have a look at another example. Suppose a list of 5 topics is given in an essay contest. A student must select one of the topics on which to write a short essay, and then select a different topic from the list for a long essay. How many ways can the topics be chosen for the two essays? In this example, there are 5 ways to choose a topic for a short essay. After choosing one for the short essay, a long essay can be written on one of the remaining 4 topics. So the number of ways to choose the topics is $5 \times 4 = 20$.

The two examples illustrate the following multiplication principle.

The Multiplication Principle

Suppose a task can be performed in m ways, and no matter how the task has been performed another task can then be performed in n ways. Then the number of ways to perform these two tasks in succession is mn.

The principle can be extended to any finite number of such successive tasks.

Example 1.

Suppose that there are 6 roads between town A and town B, and that 4 roads between town B and town C. Find the number of ways a person can drive from A to C by passing through B?

Solution

There are 6 ways to drive from A to B, and for each such way there are 4 ways to drive from B to C.

So, the number of ways to drive from A to C through B is $6 \times 4 = 24$.

Example 2.

There are four blood types, namely A, B, AB and O. Blood can also be RH(+ve) or RH(−ve). A blood donor can be classified as either male or female. How many different possible ways can a donor have his or her blood labeled?

Solution

There are 4 possibilities for the blood type, 2 possibilities for the RH factor and 2 possibilities for the gender of the donor.

So, the number of ways to label is $4 \times 2 \times 2 = 16$.

Example 3.

There are 3 picture nails on a wall. If there are 5 different pictures and each nail can hold only one picture, in how many different ways can the pictures be hung on all the nails?

Solution

Any one of the 5 pictures can be chosen for the first nail, then any one from the remaining four for the second, and finally any one from the remaining three for the third.

The number of ways to choose the pictures for individual nails are 5, 4 and 3 respectively.

So, the number of ways to hang the pictures is $5 \times 4 \times 3 = 60$.

Multiplication principle has been applied in solving counting problems in above examples. To introduce another counting principle, consider the following example.

Suppose a manufacturer makes shirts in 5 different sizes S, M, L, XL and XXL. Shirts having sizes S, M and L are made in 4 different designs, while those having sizes XL and XXL are in 3 different designs. If each design has 5 different color schemes, how may different types of shirts are possible to make?

We need to consider the two cases as follows:

(i) Shirts have sizes S, M or L. (ii) Shirts have sizes XL or XXL.

For the first case, the number of types is $3 \times 4 \times 5 = 60$.

For the second case, the number of types is $2 \times 3 \times 5 = 30$.

So the total number of types must be $60 + 30 = 90$.

Notice that the two cases we mentioned above have no arrangement in common and so these are said to be ***disjoint***. Moreover, every possible arrangement falls into some of these cases. The principle we applied above can generally be stated as follows:

The Addition Principle

Suppose there are k pairwise disjoint cases, into which the elements in a counting problem fall. If there are n_1 elements in the first case, n_2 elements in the second case, n_3 elements in the third case, ..., and n_k elements in the k^{th} case, then the number of elements in the problem is

$$n_1 + n_2 + n_3 + \cdots + n_k .$$

Example 4.

How many different numbers can be formed using the digits 3, 5, 6, 8 and 9 in such a way that the numbers contain two or three digits without any repetition ?

Solution

The two-digit numbers can be formed in $5 \times 4 = 20$ ways.

The three-digit numbers can be formed in $5 \times 4 \times 3 = 60$ ways.

So, there are $20 + 60 = 80$ different numbers.

Example 5.

How many integers, having digit 5 only once, are there between 0 and 100 ?

Solution

The integers, that we want to form, must be one of the followings:

(i) One-digit integer, namely 5

(ii) Two-digit integers having 5 as tens' digit (ones' digit may be anyone except 5)

(iii) Two-digit integers having 5 as ones' digit (tens' digit may be anyone except 0 and 5)

For the first case, the number of integers $= 1$.

For the second case, the number of integers $= 1 \times 9 = 9$.

For the third case, the number of integers $= 8 \times 1 = 8$.

The required number of integers $= 1 + 9 + 8 = 18$.

Factorial Notation

The problem in Example 3 can be modified as follows, to introduce a notation.

There are 7 picture nails on a wall. If there are 7 different pictures and each n can hold only one picture, how many different ways can the pictures be hung on all t nails?

By the multiplication principle, the pictures can be hung in $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times$ different ways. The number is the product of the first seven positive integers and c be expressed as 7!. In general, $n!$ (n factorial or factorial of n)stands for *the prod* of the first n positive integers $n, (n-1), (n-2), \ldots, 1$ if n is a positive integer. W define 0! to be 1.

> If n is a positive integer,
> $$n! = n(n-1)(n-2)\cdots 1.$$
> $$0! = 1.$$

Notice that

$$n! = n \cdot (n-1)!$$
$$n! = n \cdot (n-1) \cdot (n-2)!$$

and so on.

For examples, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = \underbrace{4 \cdot (3 \cdot 2 \cdot 1)}_{= 4 \cdot 3!} = \underbrace{4 \cdot 3 \cdot (2 \cdot 1)}_{= 4 \cdot 3 \cdot 2!}.$

Example 6.

Evaluate: (a) $\dfrac{7!}{4! \cdot 3!}$ (b) $\dfrac{6! + 5! - 4!}{4!}$

Solution

(a) $\dfrac{7!}{4! \cdot 3!} = \dfrac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2 \cdot 1} = 35$

(b) $\dfrac{6!+5!-4!}{4!} = \dfrac{6\cdot 5\cdot 4!+5\cdot 4!-4!}{4!} = \dfrac{(30+5-1)\cdot 4!}{4!} = 34$

Example 7.

Express $\dfrac{13\cdot 12\cdot 11}{3\cdot 2}$ in factorial form.

Solution $\dfrac{13\cdot 12\cdot 11}{3\cdot 2} = \dfrac{13\cdot 12\cdot 11\cdot 10!}{(3\cdot 2\cdot 1)\cdot 10!} = \dfrac{13!}{3!\cdot 10!}$

Exercise 5.1

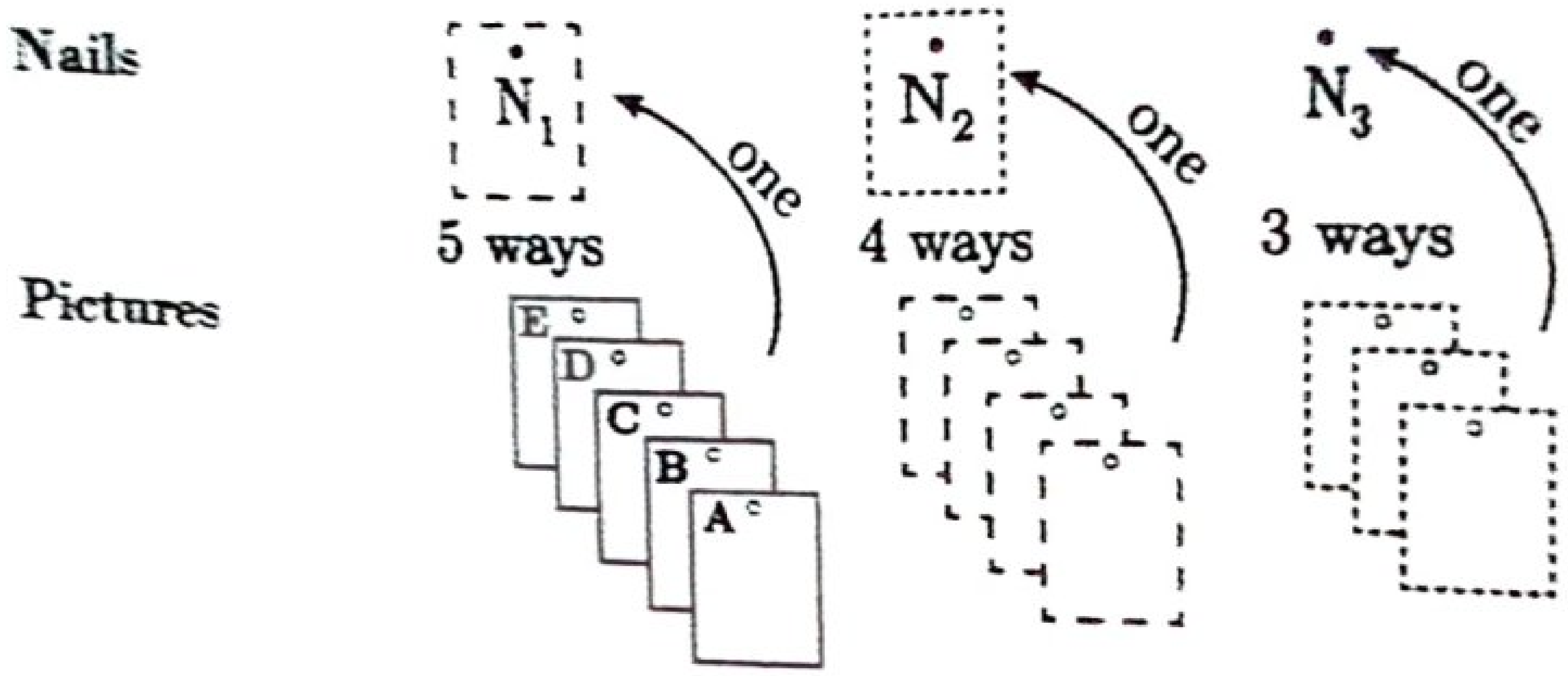
1. A store sells men's wear. It has 6 kinds of shirts, 4 kinds of pants and 3 kinds of coats. If a man wants to buy a shirt, a pant and a coat, in how many ways can this be done? (Assume that any choice meets his requirement.)
2. A television news director wishes to use three of the 7 news stories on an evening show. How many possible ways can the program be set up, if the three stories are to be classified as the lead story, the second story and the closing story?
3. If a garage door opener has a 10-digit keypad, containing 0, 1, 2, . . ., 9, and the code to open the door must be a 4-digit code, how many codes are possible to create?
4. There are 3 restaurants X, Y and Z, we can go for lunch, in a certain area. Suppose that no two restaurants have the same menu, and the numbers of choices for appetizer, main dish and dessert available are shown in the table. Find the number of ways to have a lunch, consisting of an appetizer, a main dish and a dessert.

Restaurants	No. of Appetizers	No. of Main dishes	No. of Desserts
X	4	10	2
Y	3	8	5
Z	5	12	3

5. A registration code consists of two of the 12 different capital letters A, B, C, ..., L, followed by one of the ten digits 0, 1, 2, ..., 9, for example ID5. How many codes are possible to generate:
   (a) if the repetition of the letters is allowed?
   (b) if the repetition of the letters is not allowed?
   (c) if two letters in the codes must be the same?
   (d) if two letters are different, but must be both vowels or both consonants?
6. (a) Express $\dfrac{1}{5\cdot 4\cdot 3}$ in factorial form. (b) Evaluate $2\cdot 9! + 82\cdot 8!$.
7. Prove that $\dfrac{1}{5!}+\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{50}{7!}$.

5.2 Permutations

In Example 3 in Section 5.1, we consider the arrangements of some pictures on a wall. In that problem, there are 5 different pictures (say A, B, C, D and E) and 3 nails (say N_1, N_2 and N_3).



We have known that there are $5 \cdot 4 \cdot 3$ ways to hang the pictures. If the pictures C, E and A are hung on the nails in that order, we can expressed it as CEA. We can notice that AEC and ACE are *in different orders.*

Definition. A ***permutation*** is an arrangement of the objects *in a specific order*. If there are n objects, then a permutation of any such r objects, with $r \le n$, is said to be *a permutation of n objects taken r at a time*. The number of permutations of n distinct objects taken r at a time is usually denoted by ${}^nP_r$.

In the above example, the number of ways to hang the pictures is the number of permutations of 5 pictures taken 3 at a time, and so we write

$${}^5P_3 = \underbrace{5 \cdot 4 \cdot 3}_{3 \text{ factors}}$$

- If n and r are positive integers with $r \le n$,
$${}^nP_r = \underbrace{n(n-1)(n-2)\cdots(n-(r-1))}_{r \text{ factors}}.$$
- We define ${}^nP_0$ to be 1, for a nonnegative integer n.

If n and r are positive integers with $r \le n$, we have

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\cdots(n-(r-1)) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \end{aligned}$$

Generally, for integers n and r with $0 \le r \le n$,

$${}^nP_r = \frac{n!}{(n-r)!}.$$

Notice that ${}^nP_n = n!$.

Example 8.

Evaluate ${}^{10}P_5 + {}^{10}P_0$.

Solution

$$ {}^{10}P_5 + {}^{10}P_0 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 + 1 = 30240 + 1 = 30241 $$

Example 9.

Solve the equations for n.

(a) ${}^{n}P_2 = 9n$ (b) ${}^{n}P_3 = 12 \cdot {}^{n}P_2$.

Solution

(a)

$$ \begin{aligned} {}^{n}P_2 &= 9n \\ n \geq 2 \quad \text{and} \quad n(n-1) &= 9n \\ n-1 &= 9 \\ n &= 10 \end{aligned} $$

(b)

$$ \begin{aligned} {}^{n}P_3 &= 12 \cdot {}^{n}P_2 \\ n \geq 3 \quad \text{and} \quad n(n-1)(n-2) &= 12\,n(n-1) \\ n-2 &= 12 \\ n &= 14 \end{aligned} $$

Example 10.

In how many ways can a president, a treasurer and a secretary for a committee be selected from a group of 15 people?

Solution

The number of ways to select people for three different positions (ranks) is the numbe of permutations of 15 people taken 3 at a time, and hence it is

$$ {}^{15}P_3 = 15 \cdot 14 \cdot 13 = 2730. $$

Example 11.

In how many ways can all the letters of the word PENCIL be arranged, withou repeating any letters?

Solution

There are 6 distinct letters. So the number of ways to arrange the letters is

$$ {}^{6}P_6 = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720. $$

Example 12.
There are 2 buses, which have 5 and 4 vacant seats respectively, and 4 people at a stop. In how many ways can all these people be seated on either of the buses, bu both?

Solution
The first bus has 5 vacant seats, so the number of ways to be seated there is

$$^5P_4 = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

The second bus has 4 vacant seats, so the number of ways to be seated there is

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Therefore, the required number of ways is $120 + 24 = 144$.

Example 13.
In how many ways can 6 different books be arranged along a line on a shelf if o the books is a dictionary and it must be at one end?

Solution
The dictionary must be fixed at the 1st or the 6th position in the arrangements.

If the dictionary is in the 1st position, the number of ways to arrange all the o books in remaining positions is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

If the dictionary is in the 6th position, the number of ways to arrange all the o books in remaining positions is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

So, the required number of ways $= (1 \times 120) + (120 \times 1) = 240$.

Exercise 5.2

1. Solve the equations for n. (a) $^nP_2 = 42$ (b) $^nP_3 = 9 \cdot {}^nP$
2. A newspaper has 14 reporters available to cover 3 different stories. In how m ways can the reporters be assigned to cover the stories, if no reporter ca assigned to cover more than one story?
3. Suppose we have to make a signal by choosing 4 different flags out of 9 diffe coloured flags and arranging them in a row. How many different signals ca do?
4. The manager of 4 movie theaters is deciding which of 12 available movies to sh The theaters have different seating capacities. How many ways can he sho different movies in the theaters at the same time?
5. A classroom has two rows of eight seats each. There are 10 students, 5 of wh want to sit in the front row, 4 want to sit in the back row and the remai student can sit in any seat. In how many ways can the students be seated

3.3 Combinations

Definition. A *combination* is a selection, in which the order does not matter, of objects from a collection. The number of *combinations of n distinct objects taken r at a time* is denoted by ${}^nC_r$.

The number of permutations of objects is related to the number of corresponding combinations. For example, the combinations of 4 objects A, B, C and D taken 3 at a time and the corresponding permutations are listed in the following table.

Combinations	Permutations					
{A, B, C}	ABC	ACB	BAC	BCA	CAB	CBA
{A, B, D}	ABD	ADB	BAD	BDA	DAB	DBA
{A, C, D}	ACD	ADC	CAD	CDA	DAC	DCA
{B, C, D}	BCD	BDC	CBD	CDB	DBC	DCB

We can see from the table that each combination gives rise to $3!(=6)$ permutations. Since there are ${}^4C_3$ combinations, the number of permutations must be ${}^4C_3 \cdot 3!$. Thus, we have

$${}^4C_3 \times 3! = {}^4P_3$$

Generally, for integers n and r with $0 \le r \le n$,

$${}^nC_r \times r! = {}^nP_r$$

If n and r are integers with $0 \le r \le n$, then

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}$$

Example 14.

In how many ways can a committee of 4 people be selected from a group of 10 people?

Solution.

The number of ways is ${}^{10}C_4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$

Example 15.

Evaluate ${}^{21}C_1$, ${}^{21}C_{21}$, ${}^{21}C_{20}$ and ${}^{21}C_4$

Solution.

$${}^{21}C_1 = 21$$

$${}^{21}C_{21} = \frac{21!}{21!(21-21)!} = \frac{21!}{21!\,0!} = \frac{21!}{21!\cdot 1} = 1$$

$$^{21}C_{19} = \frac{21!}{19!(21-19)!} = \frac{21!}{19!\,2!} = \frac{21 \cdot 20 \cdot 19!}{19! \cdot 2 \cdot 1} = 210$$

$$^{21}C_2 = \frac{21!}{2!\,(21-2)!} = \frac{21!}{2!\,19!} = \frac{21 \cdot 20 \cdot 19!}{2 \cdot 1 \cdot 19!} = 210$$

e notice that $^{21}C_1 = 21$, $^{21}C_{21} = 1$ and $^{21}C_{19} = {}^{21}C_2 = {}^{21}C_{21-19}$.

> Generally, for integers n and r with $0 \le r \le n$,
>
> $$^{n}C_1 = n, \quad {}^{n}C_n = 1 \quad \text{and} \quad {}^{n}C_r = {}^{n}C_{n-r}$$

xample 16.

music class consists of 5 piano players, 7 guitarists and 4 violinists. A piano player, 3 guitarists and 2 violinists must be chosen to play at a school how many ways can the band be chosen?

olution

om the music class,

1 piano player can be chosen in $^5C_1$ ways,

3 guitarists can be chosen in $^7C_3$ ways, and

2 violinists can be chosen in $^4C_2$ ways.

, the number of ways to choose a band is

$$^5C_1 \cdot {}^7C_3 \cdot {}^4C_2 = 5 \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} = 1050.$$

xample 17.

ppose there are 4 black cars and 7 white cars. If all the cars are distinguish w many ways can 3 cars of the same color be chosen?

olution

get the same color, the 3 cars must be all black or all white.

Out of 4 black cars, three can be chosen in $^4C_3$ ways.

Out of 7 white cars, three can be chosen in $^7C_3$ ways.

, the required number of ways is

$$^4C_3 + {}^7C_3 = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} + \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 4 + 35 = 39.$$

xample 18.

here are 6 different books. In how many ways can the books be given to 3 the youngest wants to receive 3 books, the elder 1 and the eldest 2 respecti

Solution

For the youngest child, the books can be chosen in $^{6}C_{3} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ ways.

For the elder child, the books can be chosen in $^{3}C_{1} = 3$ ways.

For the eldest child, the books can be chosen in $^{2}C_{2} = 1$ ways.

So, the required number of ways is $20 \cdot 3 \cdot 1 = 60$.

Example 19.

In how many ways can 4 fruits be selected out of 9 fruits, so as always to:

(a) include the largest fruit? (Assume that such a largest fruit exists.)
(b) exclude the smallest fruit? (Assume that such a smallest fruit exists.)

Solution

(a) The largest one is included in every selection. The number of ways only depends the choice of the other three from the remaining 8 fruits, and hence it is

$$^{8}C_{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

(b) Since the smallest one is excluded in every selection, we have to select 4 fruits from the remaining 8 fruits. So the required number ways is

$$^{8}C_{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70.$$

Exercise 5.3

1. Show that $^{12}C_{5} \cdot {}^{7}C_{4} = {}^{12}C_{4} \cdot {}^{8}C_{5}$.

2. There are 10 candle holders, which are fixed in different locations along a line and each can hold only one candle. In how many ways can 7 identical candles b put in these holders?

3. There are 3 parts in a test. Each of the first two parts contains 5 questions, bu the last part only 4. If a student must answer all from the first part, 4 and 3 questions from the second and the last parts respectively, in how many ways can this be done?

4. How many games can be played in a 9-team sport league if each team plays all other teams once?

5. How many lines are determined by 8 points, if no 3 such points are collinear How many triangles are determined by these points?

6. In how many ways can 4 fruits be selected out of 9 fruits, having different sizes so as always to include the largest fruit and exclude the smallest fruit?

5.4 Techniques for Some Counting Problems

Permutations with Repetitions

We had solved Example 11 about permutations of letters of the word PENCIL containing *no repeated letters*. Now consider the word KEENNESS. It has 8 letters consisting of one K, three E's, two N's and two S's. We want to determine the number of permutations of these 8 letters. It can be regarded as determining the number of ways to place these letters in 8 different places such that each letter occupies exactly one place.

Firstly we need to decide which place is to be occupied by the letter K, and this can be done in ${}^8C_1$ ways. Then there are 7 places left, among them we need to decide which places are to be occupied by 3 E's and this can be done in ${}^7C_3$ ways. After that, the places for 2 N's can be chosen in ${}^4C_2$ ways. Finally, there are ${}^2C_2$ ways to place 2 S's. By the multiplication principle, the number of permutations of all the letters is

$$ {}^8C_1 \cdot {}^7C_3 \cdot {}^4C_2 \cdot {}^2C_2 = \frac{8!}{1! \cdot \cancel{7!}} \cdot \frac{\cancel{7!}}{3! \cdot \cancel{4!}} \cdot \frac{\cancel{4!}}{2! \cdot 2!} \cdot 1 = \frac{8!}{1!\,3!\,2!\,2!}. $$

We can generalize the above result as follows.

Suppose we are given n objects, in which n_1 objects are of the first kind, n_2 objects are of the second kind, n_3 objects are of the third kind, ..., and n_r objects are of the r^{th} kind, such that $n_1+n_2+\cdots+n_r = n$. Then the number of permutations of all these n objects is $\dfrac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$

Example 20.
In how many ways can a permutation of all the letters of the word EXCELLENCE be formed?

Solution

In the word EXCELLENCE, there are 10 letters consisting of four E's, one X, two C's, two L's and one N. So number of ways is

$$ \frac{10!}{4!\,1!\,2!\,2!\,1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 1 \cdot (2 \cdot 1) \cdot (2 \cdot 1) \cdot 1} = 10 \cdot 9 \cdot 2 \cdot 7 \cdot 6 \cdot 5 = 37800. $$

The Exclusion Principle

The exclusion principle is a way to count what you are interested in by first counting what you are not interested in. This is often needed for counting where a certain property is prohibited (not allowed).

The Exclusion Principle
Count what you are not interested in and subtract it from the total.

Consider 5-digit codes containing each of the digits 1, 2, 3, 4, 5 exactly once. We ave known that the codes can be formed in 5! ways. Suppose we want to know the umber of those codes, *not ending in* 25. On contrary, we count the number of the odes ending in 25. The number of these codes is 3!, since the last two digits are fixed. ubtract it from the number of all the codes, in which the restriction for the last two igits is neglected. The resulting number 5! − 3! = 114 is the number of codes not nding in 25.

Example 21.

How many permutations are there of the letters of the word PROGRAM, if they do not end in: (a) 2R's? (b) MAP?

Solution

In the word PROGRAM, there are 7 letters consisting of one P, two R's, one O, one G, one A and one M, so the number of permutations of the letters is

$$\frac{7!}{2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 2520.$$

(*Hereafter the factor* 1! *may be omitted in the denominators of such fractions.*)

(a) The number of permutations ending in two R's is 5! = 120.
since the two R's are fixed at the end.
The number of permutations not ending in two R's = 2520 − 120 = 2400.

(b) The number of permutations ending in MAP is

$$\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 12,$$

since MAP is fixed at the end, and there are two R's in the remaining 4 letters.
The number of permutations not ending in two R's = 2520 − 12 = 2508.

Counting the Subsets of a Finite Set

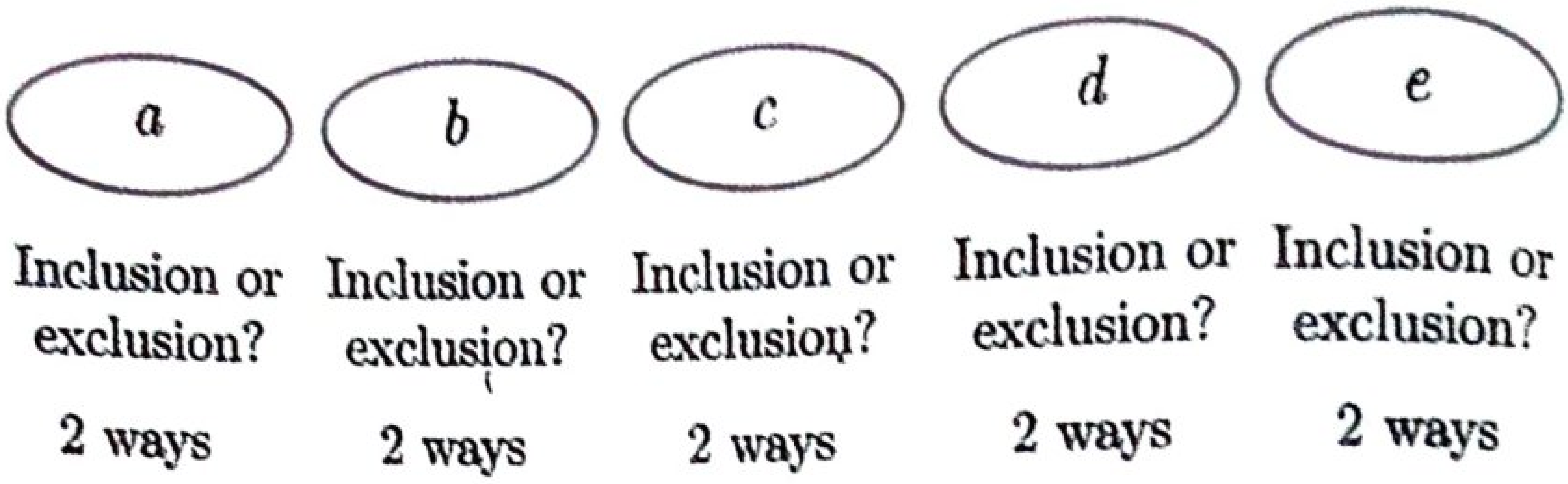
Consider a set $X = \{a, b, c, d, e\}$. A subset containing r elements, with $0 \leq r \leq 5$, can be regarded as a combination of 5 elements taking r at a time. Thus there are ${}^5C_r$ subsets containing r elements. For example, there are ${}^5C_4 = 5$ subsets containing 4 elements, as listed below.

$$\{a, b, c, d\},\ \{a, b, c, e\},\ \{a, b, d, e\},\ \{a, c, d, e\},\ \{b, c, d, e\}$$

So, by the addition principle, the number of all the subsets of X is given by

$${}^5C_0 + {}^5C_1 + {}^5C_2 + \cdots + {}^5C_5.$$

On the other hand, to form a subset, we have to decide which elements of X should be included or excluded in that subset. There are two choices for each element of X.


a
b
c
d
e
Inclusion or exclusion?
2 ways
Inclusion or exclusion?
2 ways
Inclusion or exclusion?
2 ways
Inclusion or exclusion?
2 ways
Inclusion or exclusion?
2 ways

By the multiplication principle, the number ways to form a subset of X is

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5.$$

This means that the set X has 2^5 subsets.

We have demonstrated the two different techniques for counting the subsets of set X, and noticed that

$$ {}^5C_0 + {}^5C_1 + {}^5C_2 + \cdots + {}^5C_5 = 2^5.$$

We can generalize the above result as follows.

- If a finite set contains n elements, then it has 2^n subsets.
- If n is an integer with $n \geq 0$, then ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

Example 22.

If A is a set containing 9 distinct elements, how many subsets of A contain

(a) at most 2 elements?

(b) at least 3 elements?

Solution

(a) The number of permutations containing no element, 1 element and 2 eleme ${}^9C_0$, ${}^9C_1$ and ${}^9C_2$ respectively.

So the number of subsets containing at most 2 elements is

$$\begin{aligned} {}^9C_0 + {}^9C_1 + {}^9C_2 &= 1 + 9 + \frac{9 \cdot 8}{2 \cdot 1} \\ &= 1 + 9 + 36 \\ &= 46. \end{aligned}$$

(b) The number of all the subsets of A is $2^9 = 512$.

The number of subsets containing at least 3 elements $= 512 - 46 = 466$.

Miscellaneous Counting Problems

Example 23.

How many 4-digit even numbers, greater than 4000, can be formed using the digits 1, 2, 3, 4 and 5, without repeating any digit?

Solution

Since the numbers are greater than 4000, the first digit must be 4 or 5.

Case 1. If the first digit is 4 and the last digit is 2,

First digit	Second digit	Third digit	Last digit
4			2
4	1 or 3 or 5	any one of the remaining two	2
1 way	3 ways	2 ways	1 way

the numbers can be formed in $1 \cdot 3 \cdot 2 \cdot 1 = 6$ ways.

Case 2. If the first digit is 5 and the last digit is 2 or 4,

First digit	Second digit	Third digit	Last digit
5			2 or 4
5	any one of three digits 1 or 3 or (4 or 2)	any one of the remaining two	2 or 4
1 way	3 ways	2 ways	2 ways

the numbers can be formed in $1 \cdot 3 \cdot 2 \cdot 2 = 12$ ways.

So, the number of 4-digit even numbers greater than 4000 is $6 + 12 = 18$.

Example 24.

In how many ways can 2 different chemistry books, 4 different mathematics books and 3 different physics books be arranged in a line on a shelf if

(a) 2 chemistry books are to be placed on the left, 4 mathematics books in the middle a
3 physics books on the right?

(b) books of the same subjects are together?

Solution

(a) The number of ways to place 2 chemistry books in the left part is $2! = 2 \cdot 1 = 2$.

The number of ways to place 3 physics books in the right part is $3! = 3 \cdot 2 \cdot 1 = 6$.

The number of ways to place 4 mathematics books in the middle part is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

The required number of ways $= 2 \cdot 24 \cdot 6 = 288$.

(b) The 3 subject wise groups can be placed in $3! = 6$ ways, and in each such arrangement the books can be placed in 288 ways (as in part (a)).

The required number of ways to arrange books of the same subject together $= 6 \cdot 288 = 1728$.

Example 25.

How many permutations of the letters S, U, N, D, A, Y are there if the two vowels are placed together?

Solution

If the two vowels U and A unite to form a new single letter, then the number of permutations of the letters is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Again, U and A can be arranged in 2 different ways as UA and AU in each such permutation.

So the required number of permutations is $120 \cdot 2 = 240$.

Example 26.

Find the number of permutations of all the letters of the word INTERNET in such a way that there are exactly 4 letters between the two T's.

Solution

The letters must be arranged in one of the three forms:

$$\mathrm{T}\star\star\star\star\mathrm{T}\star\star \qquad \star\mathrm{T}\star\star\star\star\mathrm{T}\star \qquad \star\star\mathrm{T}\star\star\star\star\mathrm{T}$$

in which the 6 letters (2 N's, 2 E's, 1 I and 1 R) other than 2 T's are to be put in star ($\star$) positions. For each such form, the number of ways to put the six letters in star ($\star$) positions is

$$\frac{6!}{2! \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2! \cdot 2 \cdot 1} = 180.$$

So the required number of permutations is $3 \cdot 180 = 540$.

Exercise 5.4

1. Find the number of permutations of the letters of each of the following words.
   (i) INFINITY (ii) PINEAPPLE (iii) ACCEPTABLE

2. Passwords for a mobile payment system are to consist of 6 digits from a set of digits 0, 1, 2, . . . , 9 and assume that there is no other restriction. How many such passwords have repeated digits?

3. A building has 10 windows in front. What is the number of signals that can be given, by having one or more of the windows open?

4. At a burger shop, 3 different types of buns, 7 types of cheeses and 5 types of vegetables are available. Assume that a burger may contain no cheese or no vegetable. In how many ways can different types of burgers be ordered if a person must have a bun, and may have at most 2 types of cheeses and any number of types vegetables.

5. How many different 4-digit codes can be formed using all the digits 0, 1, 2, 3 if
   (i) there is no restriction?
   (ii) repetition is not allowed?
   (iii) repetition is not allowed, and 0 is either the first or the last digit?

6. Using the letters of the word EQUATION without repetitions, how many 4-letter codes can be formed:
   (i) starting with T and ending with N?
   (ii) starting and ending with a consonant?
   (iii) with vowels only?
   (iv) if it contains 3 consonants?

7. Three brothers and three sisters are lining up to be photographed. How many arrangements are there
   (i) with 3 sisters standing together?
   (ii) if brothers and sisters are in alternating positions?

8. How many permutations of the letters H, E, X, A, G, O, N are there if
   (i) the 3 vowels are placed together?
   (ii) the 3 vowels are not placed together?
   (iii) consonants and vowels do not appear alternately?

9. Find the number of permutations of all the letters of the word PENGUIN in such a way that there are exactly 3 letters between 2 N's.

10. Find the number of permutations of all the letters of the word STRESSLESS in such a way that there are exactly 5 letters between:
    (i) 2 E's. (ii) 2 S's.

Chapter 6

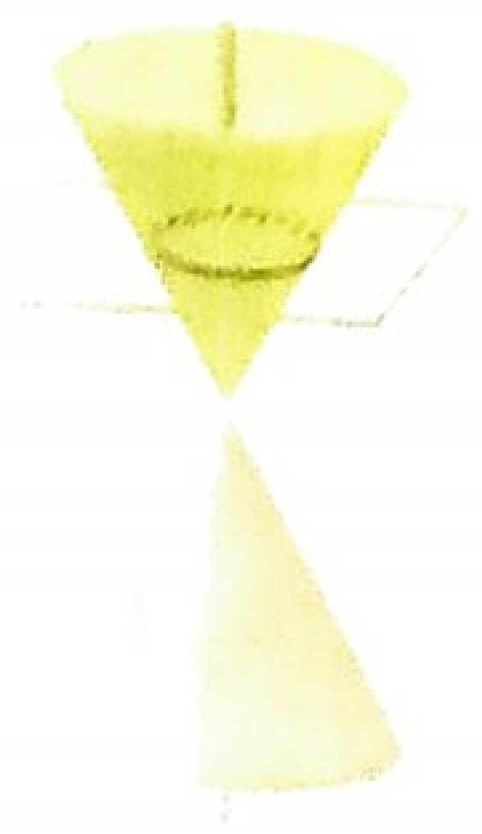
Conic Sections

In this chapter, we will study the basic properties and equations of circles and parabolas.

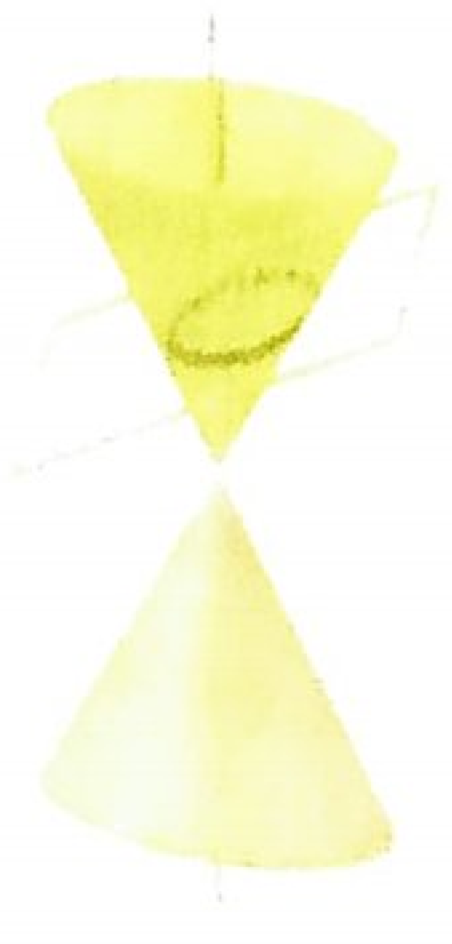
6.1 Introduction

The curves that can be obtained by the intersections of a plane with a double right-circular cone are called **conics** or **conic sections**. The cone consists of two parts, called nappes that intersect at the vertex. The most important of conic sections or simply conics are circles, ellipses, parabolas and hyperbolas.

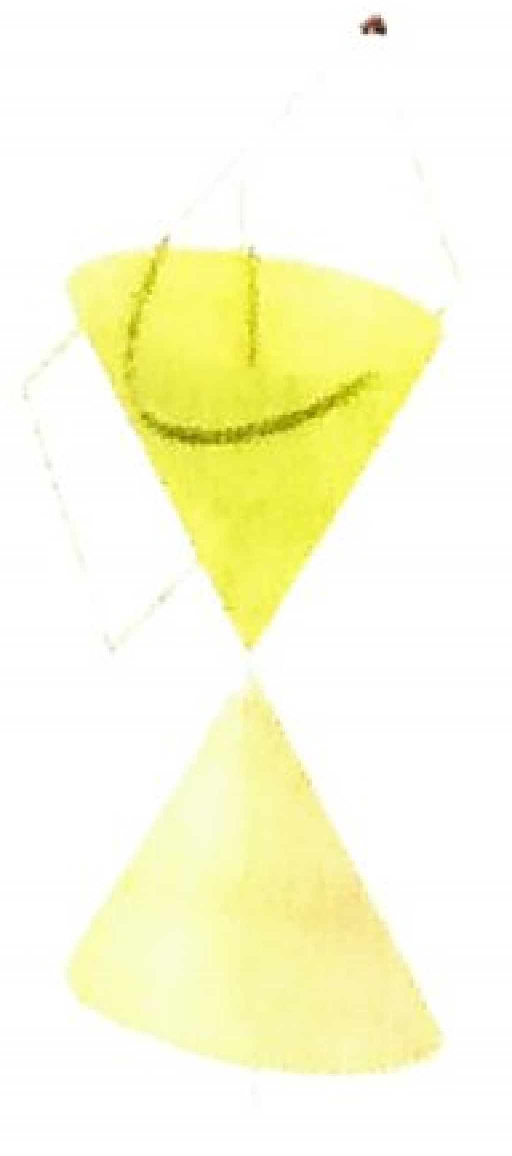
Double right circular cone



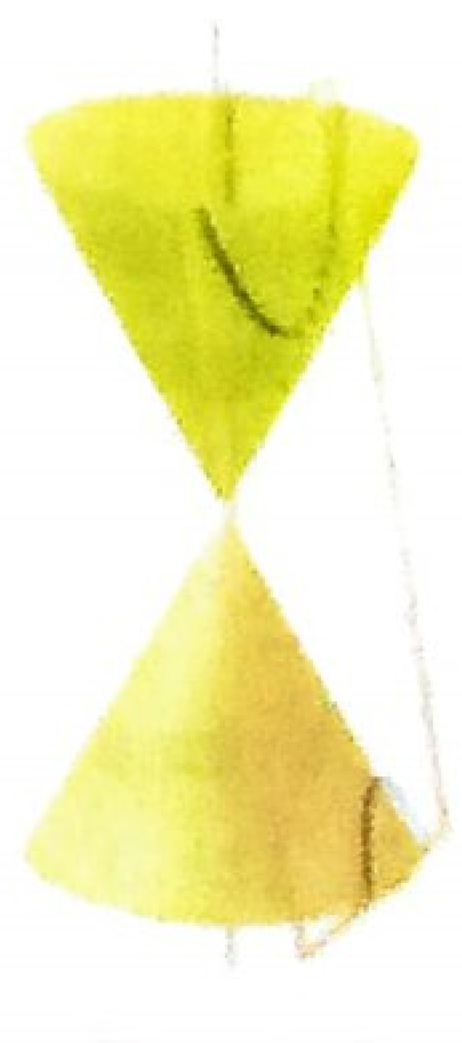
Circle



Ellipse



Parabola



Hyperbola

A **circle** is obtained by cutting a cone with a plane that is perpendicular to the axis and does not contain the vertex. If the cutting plane is tilted slightly and intersects only one nappe, the resulting intersection is an **ellipse**. If the cutting plane is tilted still further so that it is parallel to a line on the surface of a cone, but intersects only ne nappe, the resulting intersection is a **parabola**. Finally, if the plane intersects oth nappes, but does not contain the vertex, the resulting intersection is a **hyperbola**.

All conics can be written in the form of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The equations for each of these conics can be written in a standard form which can tell the different types of conics without having to graph it.

Today, properties of conic sections are used in the construction of telescopes, radar antennas, medical equipment, navigational systems, and in the determination of satellite orbits.

6.2 Circles

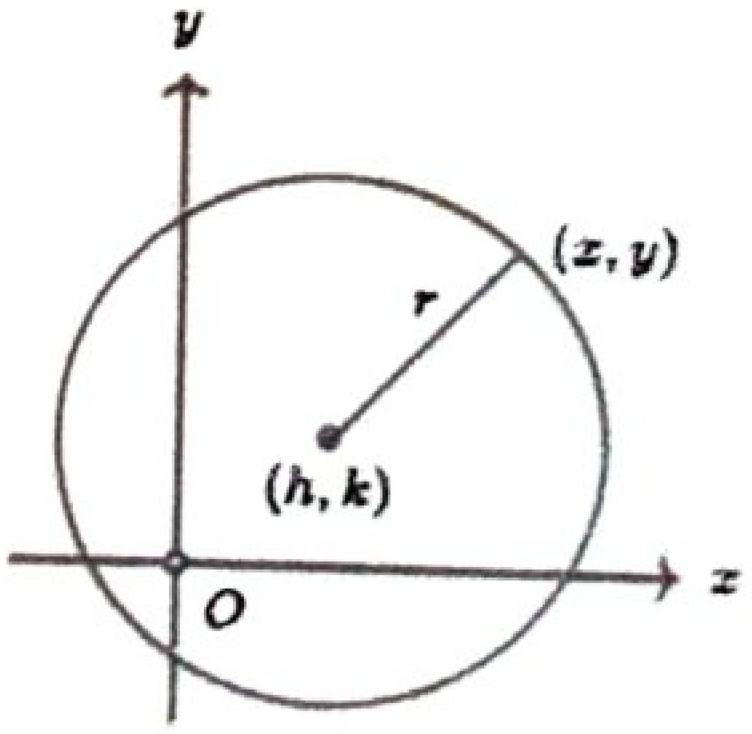
A circle is the set of all points in a plane that are equidistant from a fixed point. The fixed point is called the center of the circle and the equidistant is called the radius of the circle.

Now, we consider a circle with a radius r and the center (h, k).

Standard Form of the Equation of a Circle

If (h, k) is a fixed point in the plane, then the circle of radius r centered at (h, k) is the set of all points in the plane whose distance from (h, k) is r.

Thus, a point (x, y) will lie on this circle if and only if



$$\sqrt{(x-h)^2 + (y-k)^2} = r \qquad (1)$$

or equivalently,

$$(x-h)^2 + (y-k)^2 = r^2. \qquad (2)$$

This is called the **standard form of the equation of a circle**. We can find immediately the center and radius from the equation.
The center $= (h, k)$ and the radius $= r$.

General Form of the Equation of a Circle

Expanding the equation 2 , we get

$$x^2 + y^2 - 2hx + 2ky + h^2 + k^2 - r^2 = 0.$$

Taking $2g = -2h$, $2f = -2k$, $c = h^2 + k^2 - r^2$, the equation becomes the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This is called the **general form of the equation of a circle.**
We consider the conic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

If $A = C$ and $B = 0$, the equation becomes a circle equation.
Note that $B = 0$ means that the equation does not contain the term xy. Again, we can find the center and radius of this equation by using the technique of completing the squares.

$$\begin{aligned} x^2 + y^2 + 2gx + 2fy &= -c \\ x^2 + 2gx + g^2 + y^2 + 2fy + f^2 &= g^2 + f^2 - c \\ (x+g)^2 + (y+f)^2 &= g^2 + f^2 - c \end{aligned}$$

In this case, the center is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$.

Degenerate Cases of a Circle

Consider the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

- If $g^2 + f^2 - c > 0$, the graph is a circle with center at $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.
- If $g^2 + f^2 - c = 0$, the graph is a single point $(-g, -f)$.
- If $g^2 + f^2 - c < 0$, the equation has no real solution and consequently no graph

Example 1.

Find an equation of the circle of radius 4 centered at $(-1, 2)$ in standard form.

Solution

Since $r = 4$, $(h, k) = (-1, 2)$, the equation of the circle is

$$\begin{aligned} (x+1)^2 + (y-2)^2 &= 4^2 \\ (x+1)^2 + (y-2)^2 &= 16. \end{aligned}$$

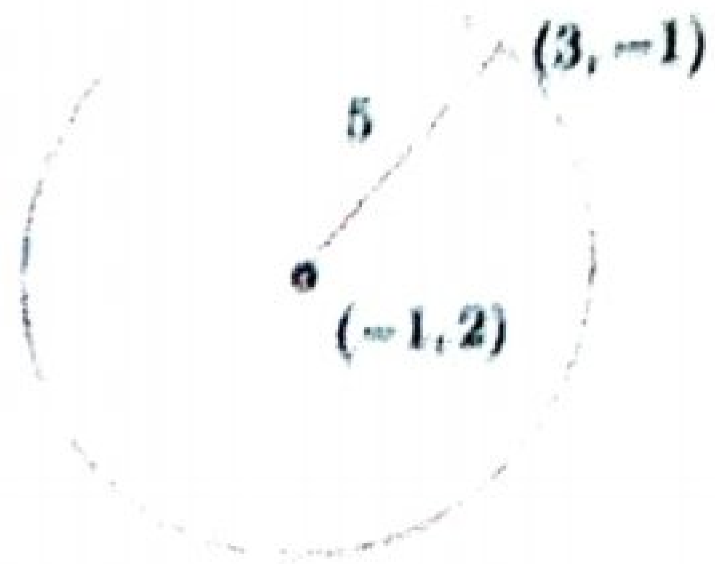
Example 2.

Find an equation for the circle with center $(-1, 2)$ that passes through $(3, -1)$.

Solution

Since $(h, k) = (-1, 2), (x_1, y_1) = (3, -1)$, we have

$$r = \sqrt{(-1-3)^2 + (2+1)^2} = 5.$$



Thus, the equation of the circle is

$$(x+1)^2 + (y-2)^2 = 5^2$$

$$x^2 + y^2 + 2x - 4y - 20 = 0.$$

Example 3.

Find the center and radius of each of the following circles.

(a) $(x-3)^2 + (y-5)^2 = 16$ (b) $(x+7)^2 + (y-2)^2 = 9$

(c) $x^2 + y^2 = 1$ (d) $x^2 + (y-3)^2 = 49$

Solution

Equation of Circle	Center (h, k)	Radius r
$(x-3)^2 + (y-5)^2 = 16$	$(3, 5)$	4
$(x+7)^2 + (y-2)^2 = 9$	$(-7, 2)$	3
$x^2 + y^2 = 1$	$(0, 0)$	1
$x^2 + (y-3)^2 = 49$	$(0, 3)$	7

The circle $x^2 + y^2 = 1$ is the **unit circle** centered at the origin and has radius 1.

Example 4.

Find the center and radius of the following circles.

(a) $x^2 + y^2 - 2x + 10y + 23 = 0$ (b) $2x^2 + 2y^2 - 24x + 47 = 0$

Solution

(a) Method 1

$$x^2 + y^2 - 2x + 10y + 23 = 0$$

$$x^2 - 2x + y^2 + 10y = -23$$

$$x^2 - 2x + 1^2 + y^2 + 10y + 5^2 = 1 + 25 - 23$$

$$(x-1)^2 + (y+5)^2 = 3$$

Therefore, the center $= (1, -5)$ and radius $= \sqrt{3}$.

Method 2

$$x^2 + y^2 - 2x + 10y + 23 = 0$$

Comparing with $\quad x^2 + y^2 + 2gx + 2fy + c = 0,$

we get $\quad 2g = -2, \quad 2f = 10, \quad c = 23,$

and so $\quad g = -1, \quad f = 5, \quad c = 23.$

Therefore, the center $= (-g, -f) = (1, -5)$ and

$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 25 - 23} = \sqrt{3}\,.$$

(b) Method 1

$$2x^2 + 2y^2 - 24x + 47 = 0$$

$$x^2 + y^2 - 12x + \frac{47}{2} = 0$$

$$x^2 - 12x + y^2 = -\frac{47}{2}$$

$$x^2 - 12x + 6^2 + y^2 = 36 - \frac{47}{2}$$

$$(x-6)^2 + y^2 = \frac{25}{2}$$

$$(x-6)^2 + y^2 = (\frac{5}{\sqrt{2}})^2$$

The center $= (6, 0)$ and radius $= \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$.

Method 2

$$2x^2 + 2y^2 - 24x + 47 = 0$$

$$x^2 + y^2 - 12x + \frac{47}{2} = 0$$

Comparing with $\quad x^2 + y^2 + 2gx + 2fy + c = 0,$

we get $\quad 2g = -12, \quad 2f = 0, \quad c = \frac{47}{2},$

and so $\quad g = -6, \quad f = 0.$

Therefore, the center $= (-g, -f) = (6, 0)$ and

$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{36 + 0 - \frac{47}{2}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

Example 5.

Describe the graphs of the following equations.

(a) $(x-1)^2+(y+4)^2=-9$

(b) $(x-1)^2+(y+4)^2=0$

(c) $x^2+y^2-4x+2y-44=0$

Solution

(a) $(x-1)^2+(y+4)^2=-9$

We see that the right hand side is less than zero, so the equation has no graph.

(b) $(x-1)^2+(y+4)^2=0$

The graph is a single point $(1,-4)$.

(c) $x^2+y^2-4x+2y-44=0$

Comparing with $x^2+y^2+2gx+2fy+c=0$,

we get $2g=-4,\quad 2f=2,\quad c=-44,$

and so $g=-2,\quad f=1,\quad c=-44.$

Since $g^2+f^2-c=4+1+44=49>0,$

the graph is a circle with center at $(2,-1)$ and radius is 7.

Exercise 6.1

1. Find the center and radius of each circle:

(a) $x^2+y^2=16$

(b) $(x-1)^2+(y-4)^2=25$

(c) $(x-3)^2+(y+5)^2=49$

(d) $(x+1)^2+y^2=1$

(e) $2x^2+2y^2-12x+8y-24=0$

(f) $x^2+y^2+x+y-\frac{1}{2}=0$

2. Find the standard equation of the circle satisfying the given conditions.

(a) center$(3,-2)$, radius$=4$.

(b) center$(2,5)$, radius$=\sqrt{5}$.

(c) center$(-4,1)$, radius$=3$.

(d) center$(-4,8)$, circle is tangent to the x-axis.

(e) center$(5,8)$, circle is tangent to the y-axis.

(f) center$(-3,-2)$, circle passes through the origin.

(g) center$(4,-5)$, circle passes through $(1,3)$.

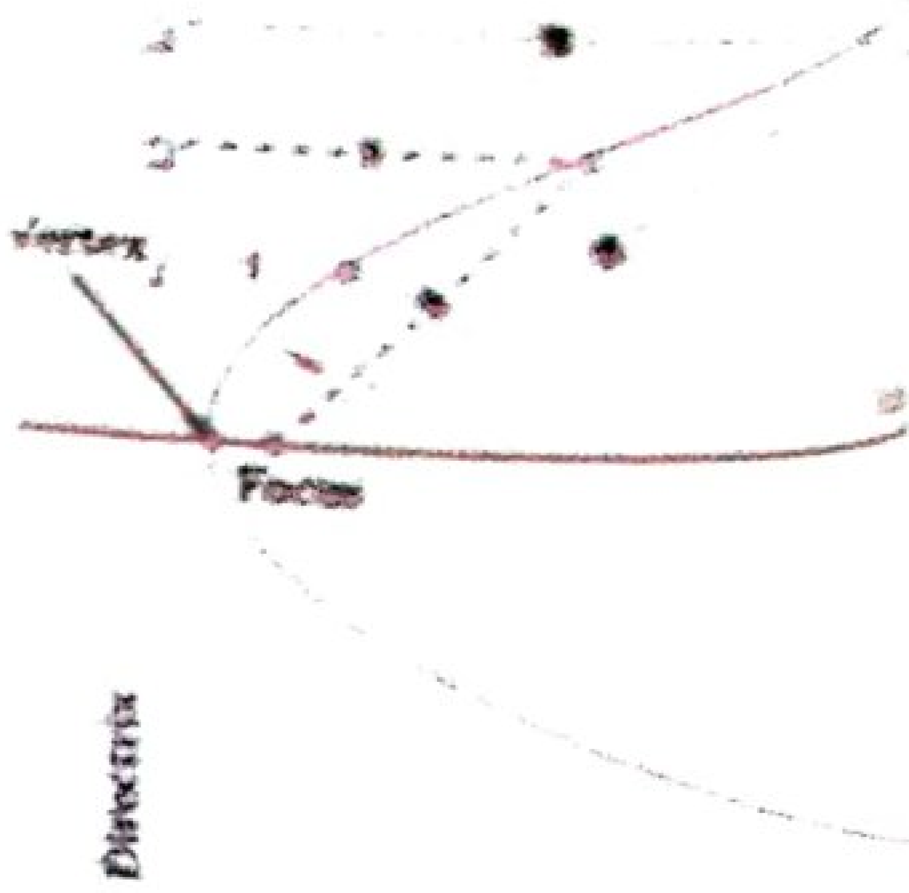
(h) diameter has end points $(6,1)$ and $(-2,3)$.

3. Determine whether each equation represents a circle, a point or no graph. equation represents a circle, find the center and radius.

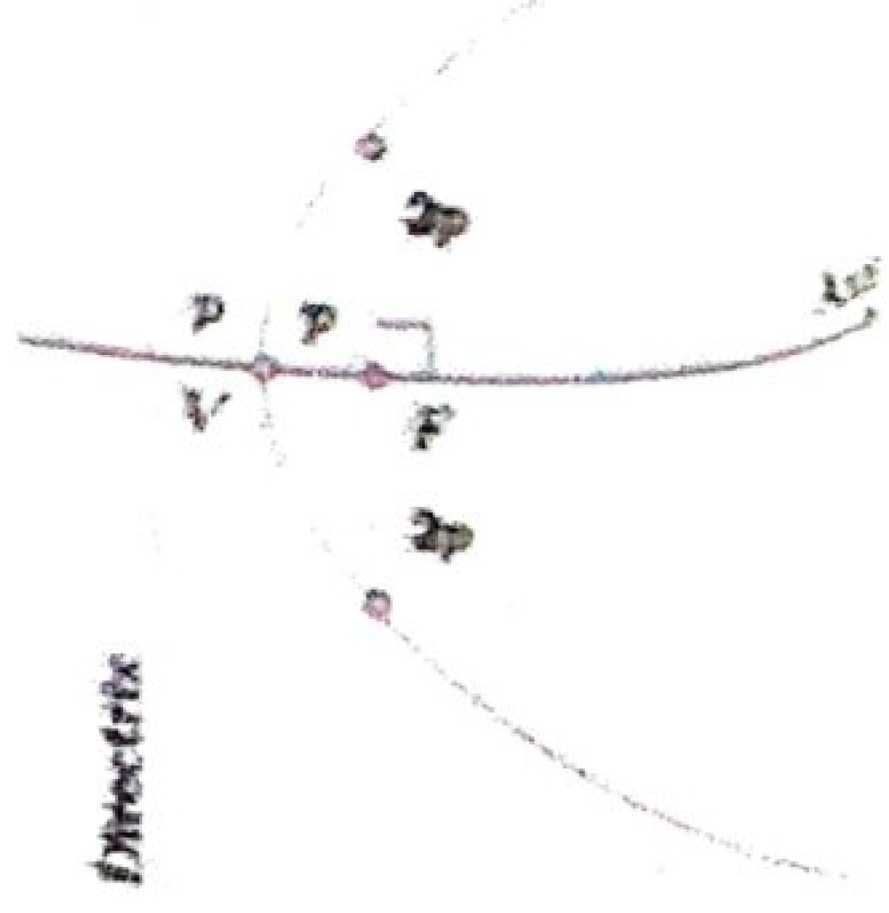
(a) $x^2 + y^2 - 6x + 2y + 9 = 0$
(b) $x^2 + y^2 - 4x - 6y + 13 = 0$
(c) $x^2 + y^2 + 10y + 26 = 0$
(d) $2x^2 + 2y^2 - 12x + 8y - 24 =$
(e) $x^2 + y^2 - 10x - 2y + 29 = 0$
(f) $16x^2 + 16y^2 + 40x + 16y - 7 =$
(g) $\frac{x^2}{4} + \frac{y^2}{4} = 1$
(h) $2x^2 + 2y^2 + 8x + 7 = 0$

5.3 The Parabola

A **parabola** is the set of all points (x, y) in the plane that are equidistant from a
ine and a fixed point which is not on the fixed line.



n this section, we will discuss the properties of the arabolas. The fixed line is called the **directrix** f the parabola and the fixed point is the **focus**. The line through the focus and perpendicular to he directrix is called the **axis** of the parabola. The oint of intersection of the axis and the parabola is alled the **vertex**. A parabola is symmetric about he axis.



Denote the distance between the focus F and the ertex V by p. The vertex is equidistant from the ocus and the directrix. Therefore, the distance etween focus and directrix is $2p$. As illustrated n the figure, the parabola passes through two f the corners of the box that extends from the ertex to the focus along the axis of symmetry and xtends $2p$ units above and $2p$ units below the axis f symmetry. The line segment passes through the ocus and is parallel to the directrix is called the **atus rectum**. The end points of the latus rectum e on the curve.

General Form of the Equation of a Parabola

From the general form of the conics equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, we begin with the case where $B=0$. In this case, the term containing xy is not present and it becomes

$$Ax^2+Cy^2+Dx+Ey+F=0.$$

Furthermore, one of A and C equals 0 becomes a parabola. We have already discussed the procedure for identifying the graph of this kind of equation; we complete the squares of the quadratic expressions in x or y, or both. Then we have

$$Ax^2+Dx+Ey+F=0$$

or

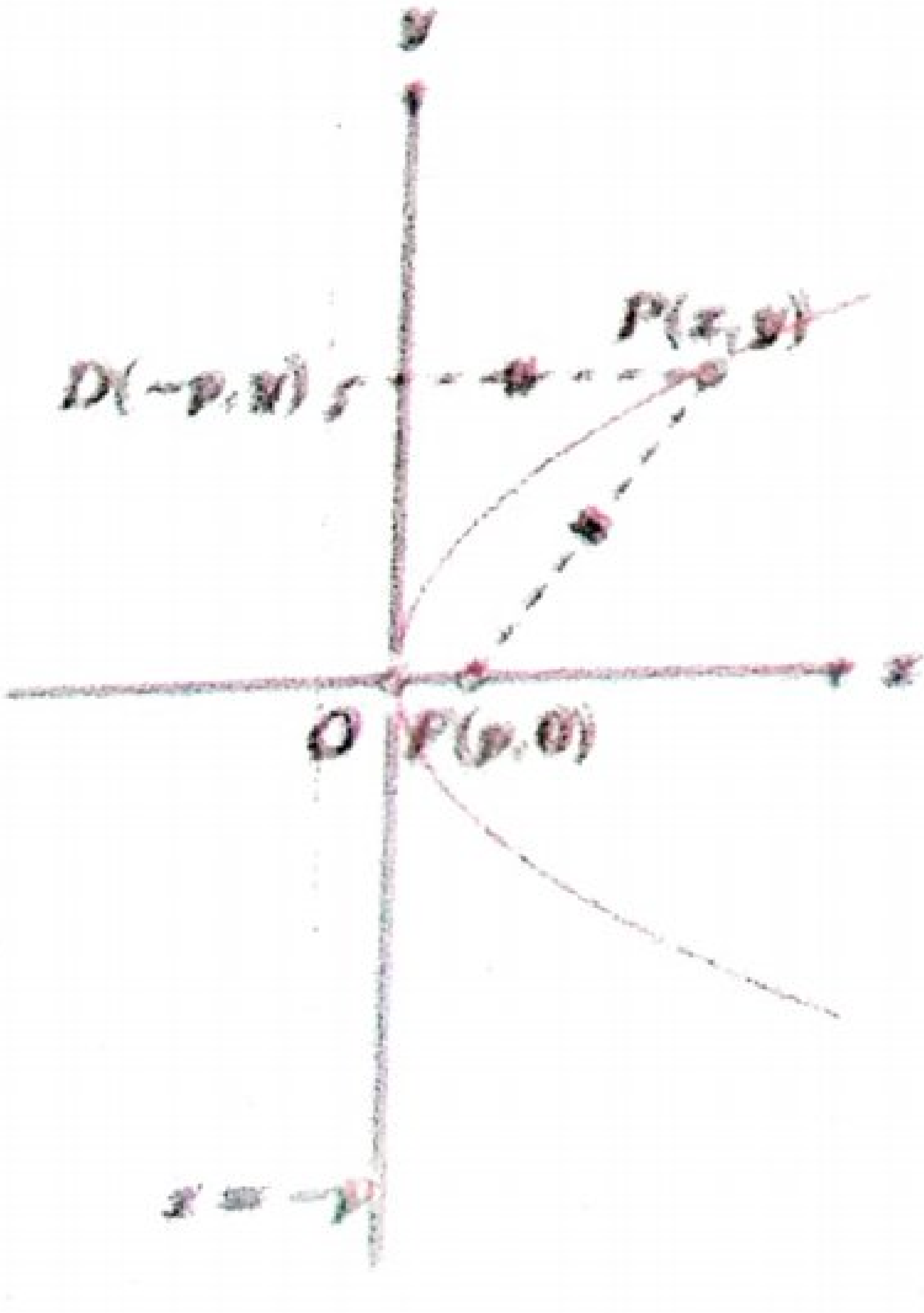
$$Cy^2+Dx+Ey+F=0.$$

Above two equations are called the **general forms of equation of a parabola** . We have learned that the graph of the quadratic function $y=ax^2+bx+c,\ a\neq 0$ is a parabola. If $a>0$. the graph opens up and otherwise, it opens down.

Equations of a Parabola: Vertex at $(0,0)$

The equation of a parabola is simplest if the coordinate axes are positioned so that the vertex is at the origin and the axis is along the x-axis or y-axis.

We will derive the equation for the parabola. Let $P(x,y)$ be any point on the parabola. Since P is equidistant from the focus $F(p,0)$ and $D(-p,y)$ is the foot of the perpendicular from P to the directrix, where $p>0$, we have $PF=PD$.



By the distance formula,

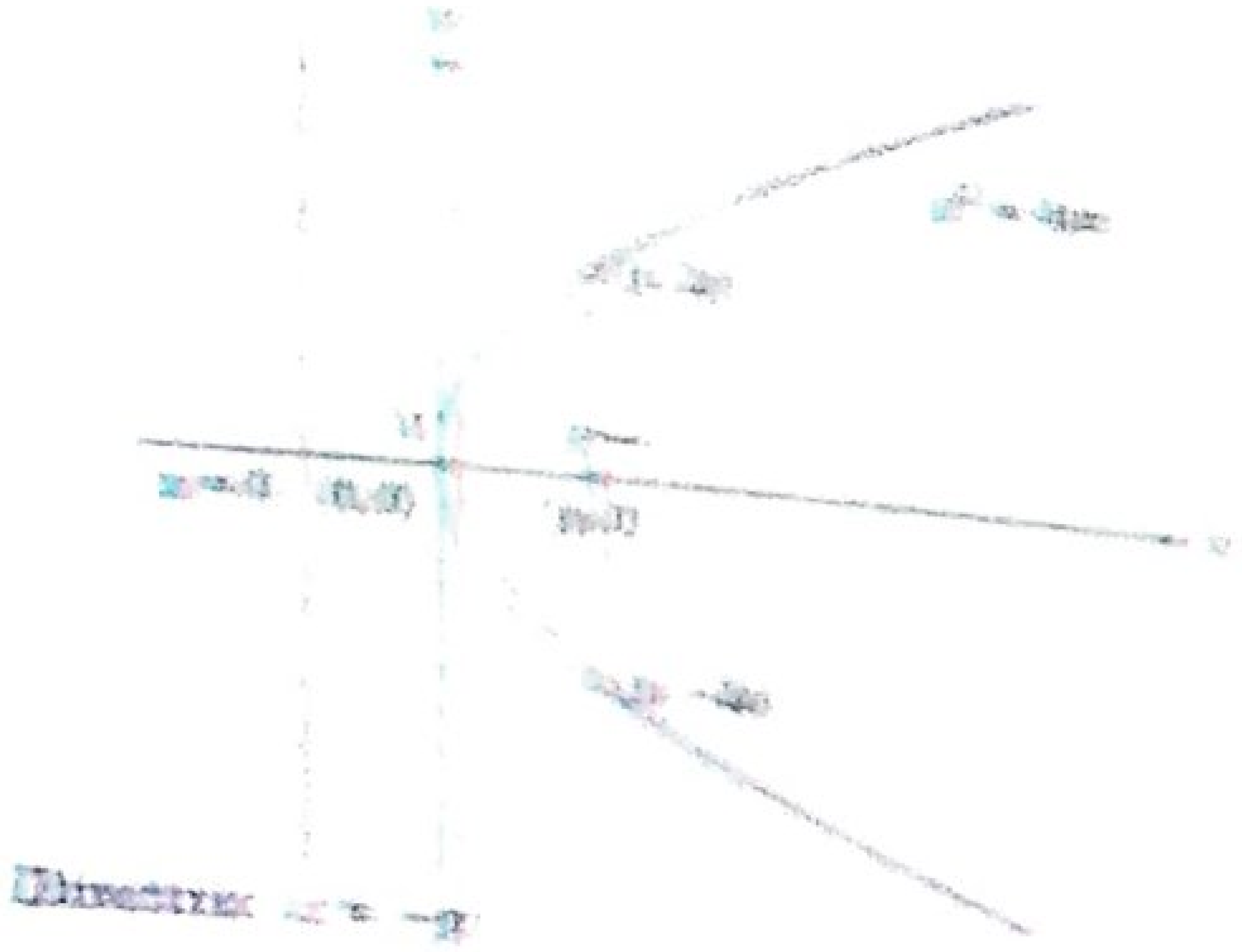
$$PF = \sqrt{(p-x)^2 + (0-y)^2},$$

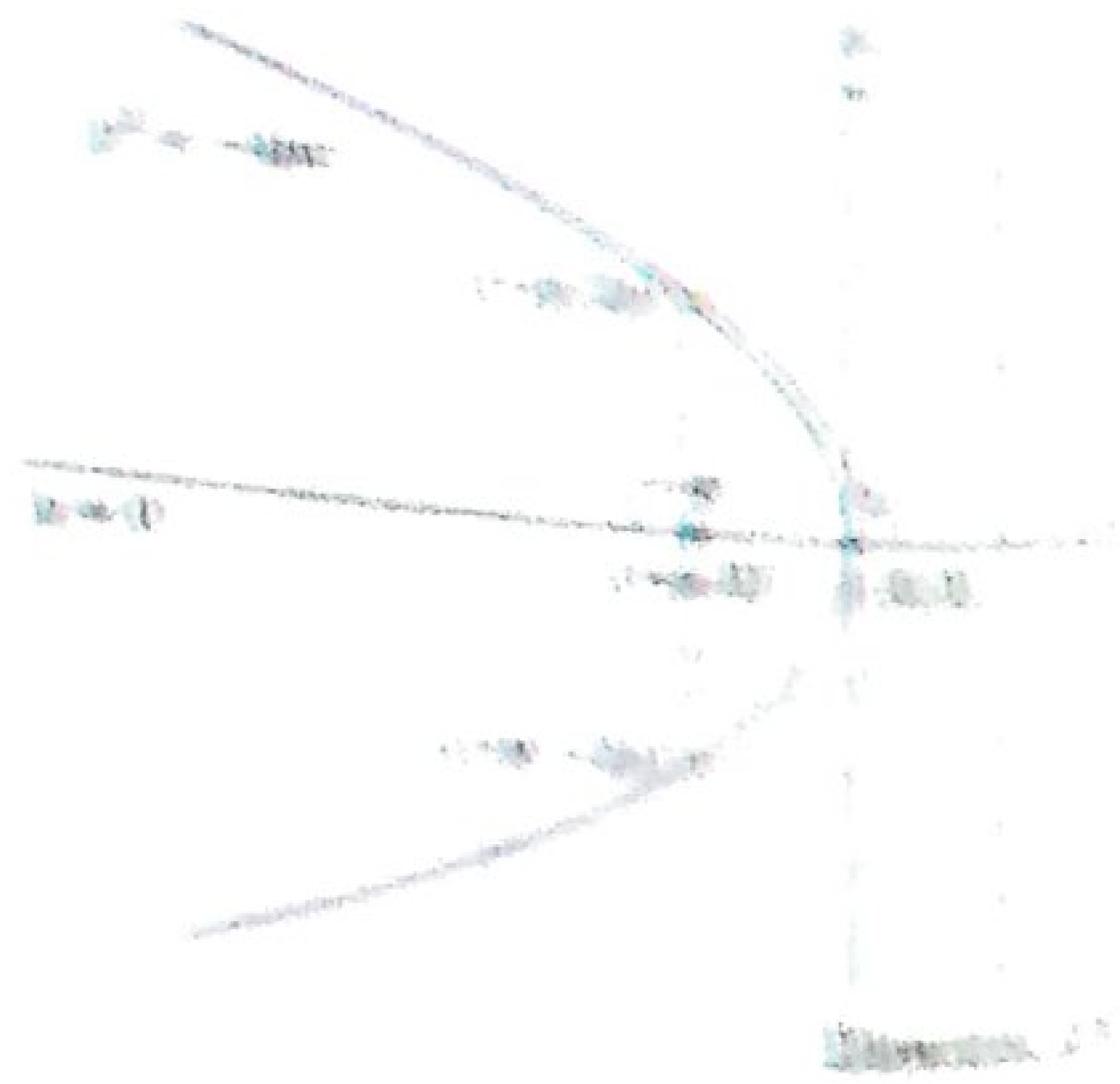
$$PD = \sqrt{(-p-x)^2 + (y-y)^2}.$$

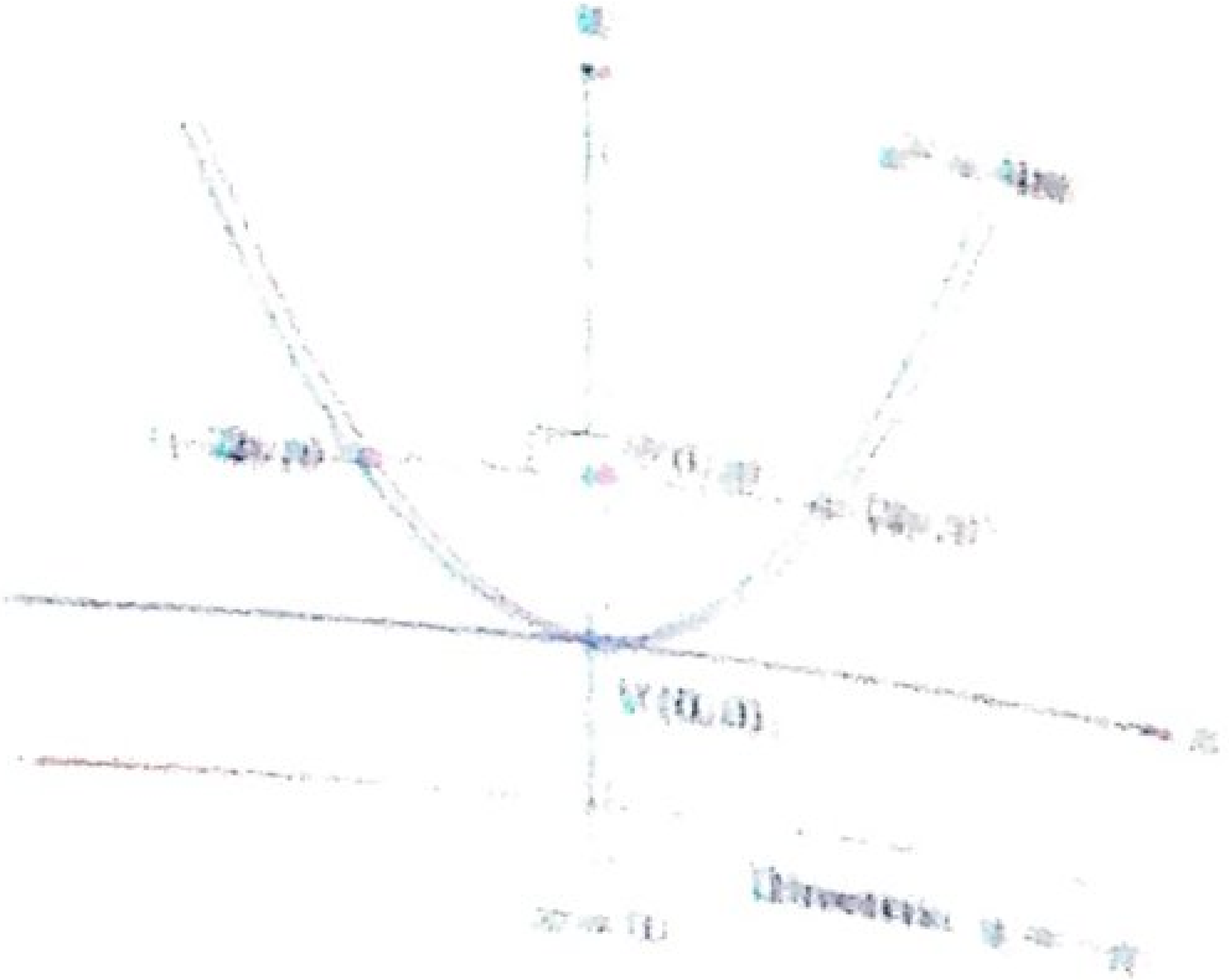
Since $PF = PD$, we have

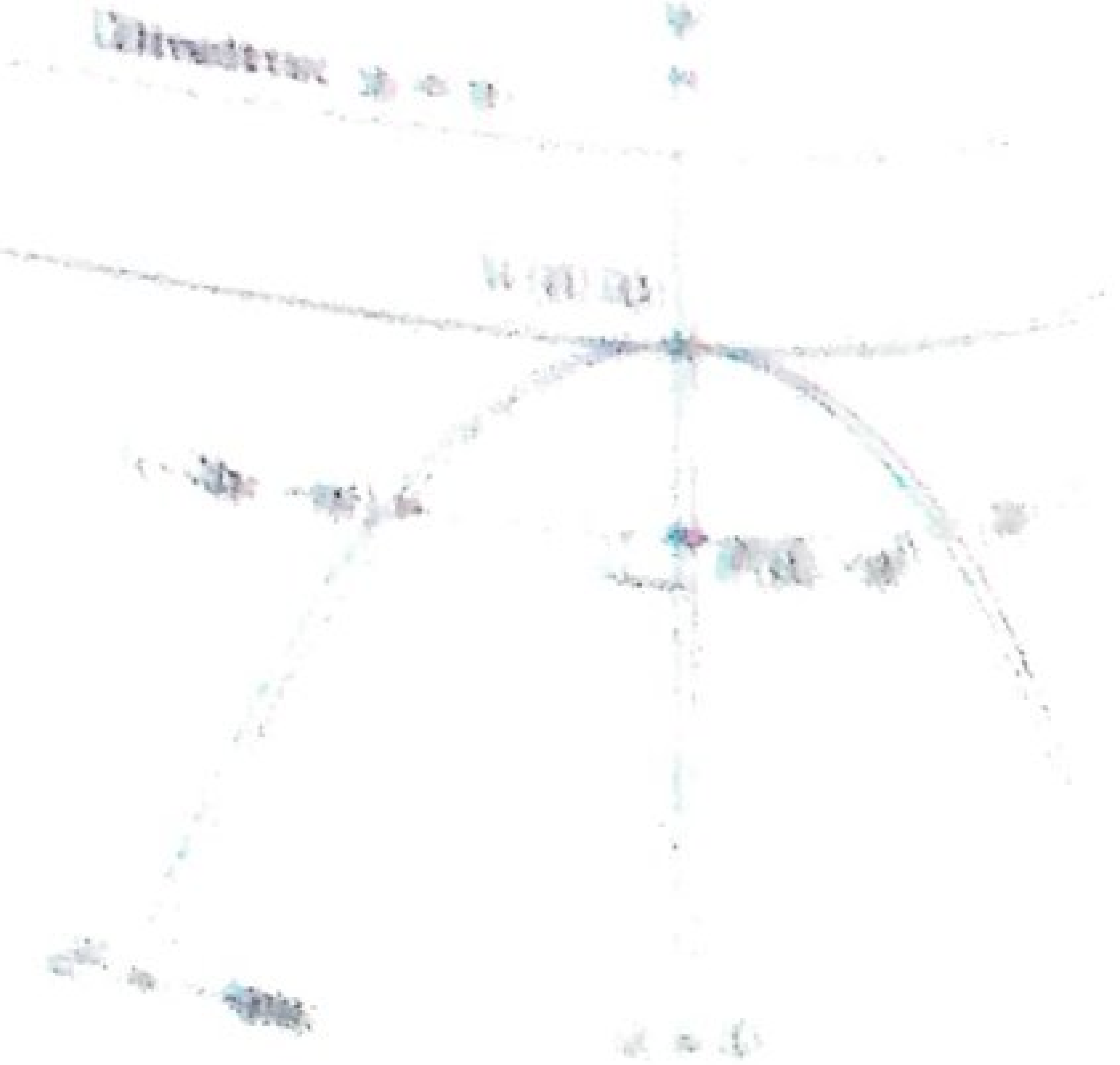
$$\begin{aligned}\sqrt{(p-x)^2 + (0-y)^2} &= \sqrt{(-p-x)^2 + (y-y)^2}\\ p^2 - 2px + x^2 + y^2 &= p^2 + 2px + x^2\\ y^2 &= 4px.\end{aligned}$$

The four possible orientations are shown in the following figures. These are called standard positions of a parabola, and the resulting equations are called the standa **forms of equation of a parabola.**


$y^2 = 4px$
Directrix $x = -p$




$x^2 = 4py$
$V(0,0)$
Directrix $y = -p$


Directrix $y = p$
$x^2 = -4py$

Equations of a Parabola:
Vertex at $(0, 0)$, Focus on Coordinate Axis, $p > 0$

Equation	Vertex	Focus	Directrix	Axis of Symmetry	Description
$y^2 = 4px$	$(0,0)$	$(p,0)$	$x = -p$	horizontal axis, $y = 0$	opens to the right
$y^2 = -4px$	$(0,0)$	$(-p,0)$	$x = p$	horizontal axis, $y = 0$	opens to the left
$x^2 = 4py$	$(0,0)$	$(0,p)$	$y = -p$	vertical axis, $x = 0$	opens up
$x^2 = -4py$	$(0,0)$	$(0,-p)$	$y = p$	vertical axis, $x = 0$	opens down

Parabolas can be graphed by using the following four steps :

- The axis of symmetry is along the x-axis if the equation has a y^2 term, and it is along the y-axis if the equation has a x^2 term.
- If the axis of symmetry is along the x-axis, then the parabola
  - opens to the right if the coefficient of x is positive.
  - opens to the left if the coefficient of x is negative.
- If the axis of symmetry is along the y-axis, then the parabola
  - opens up if the coefficient of y is positive.
  - opens down if the coefficient of y is negative.
- Determine the value of p from the standard equation and draw a box extending p units from the origin along the axis of symmetry in the direction in which the parabola opens and extending $2p$ units on each side of the axis of symmetry.

Using the above facts, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box.

Example 6.
Sketch the graphs of the parabolas, showing the vertex, focus, directrix and end points of latus rectum of each equation.

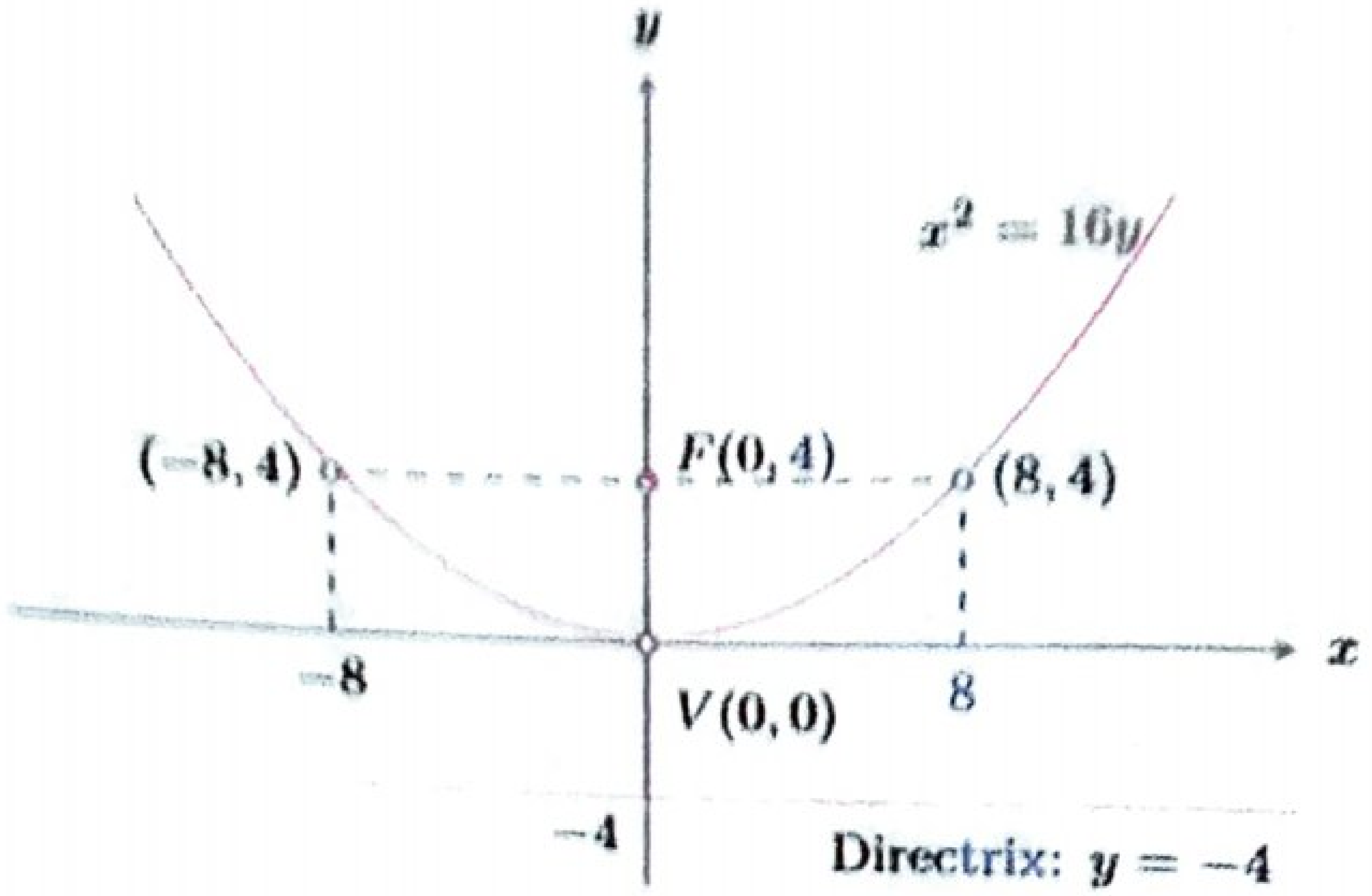
(a) $x^2 = 16y$ (b) $y^2 + 8x = 0$

Solution

(a) $x^2 = 16y$

Comparing with $x^2 = 4py$, we get $p = 4$.

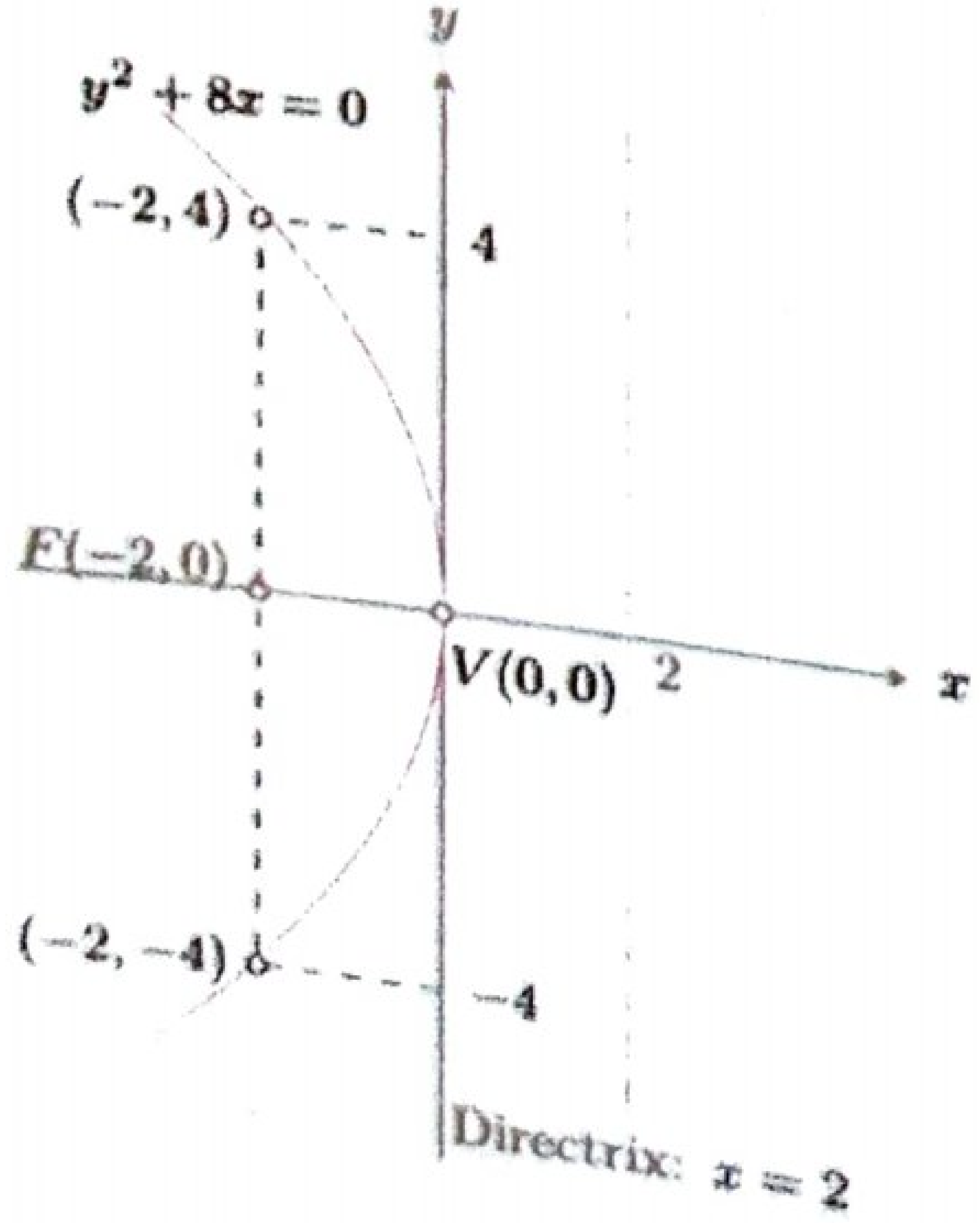
Therefore, the vertex $V = (0, 0)$, the focus $F = (0, 4)$, the directrix is $y = -4$ and end points of latus rectum are $(-8, 4), (8, 4)$.



(b) $y^2 + 8x = 0$

$y^2 = -8x$

Comparing with $y^2 = -4px$, we get $p = 2$.
Therefore, the vertex $V = (0,0)$, the focus $F(-2,0)$, the directrix is $x = 2$ and end points of latus rectum are $(-2,-4), (-2,4)$.



Example 7.

Find an equation of the parabola that is symmetric about the y-axis with its vertex at the origin, and passes through $(5, 2)$.

Solution

Since the parabola is symmetric about the y-axis, the vertex $V = (0, 0)$ and passes through $(5, 2)$, the equation of the form is

$$x^2 = 4py$$
$$5^2 = 4p(2)$$
$$4p = \frac{25}{2}.$$

Therefore, the equation of the parabola is

$$x^2 = \frac{25}{2}y.$$

Exercise 6.2

In Exercise 1 - 8, (a) sketch the parabola. (b) show the focus, vertex, directrix and end points of latus rectum.

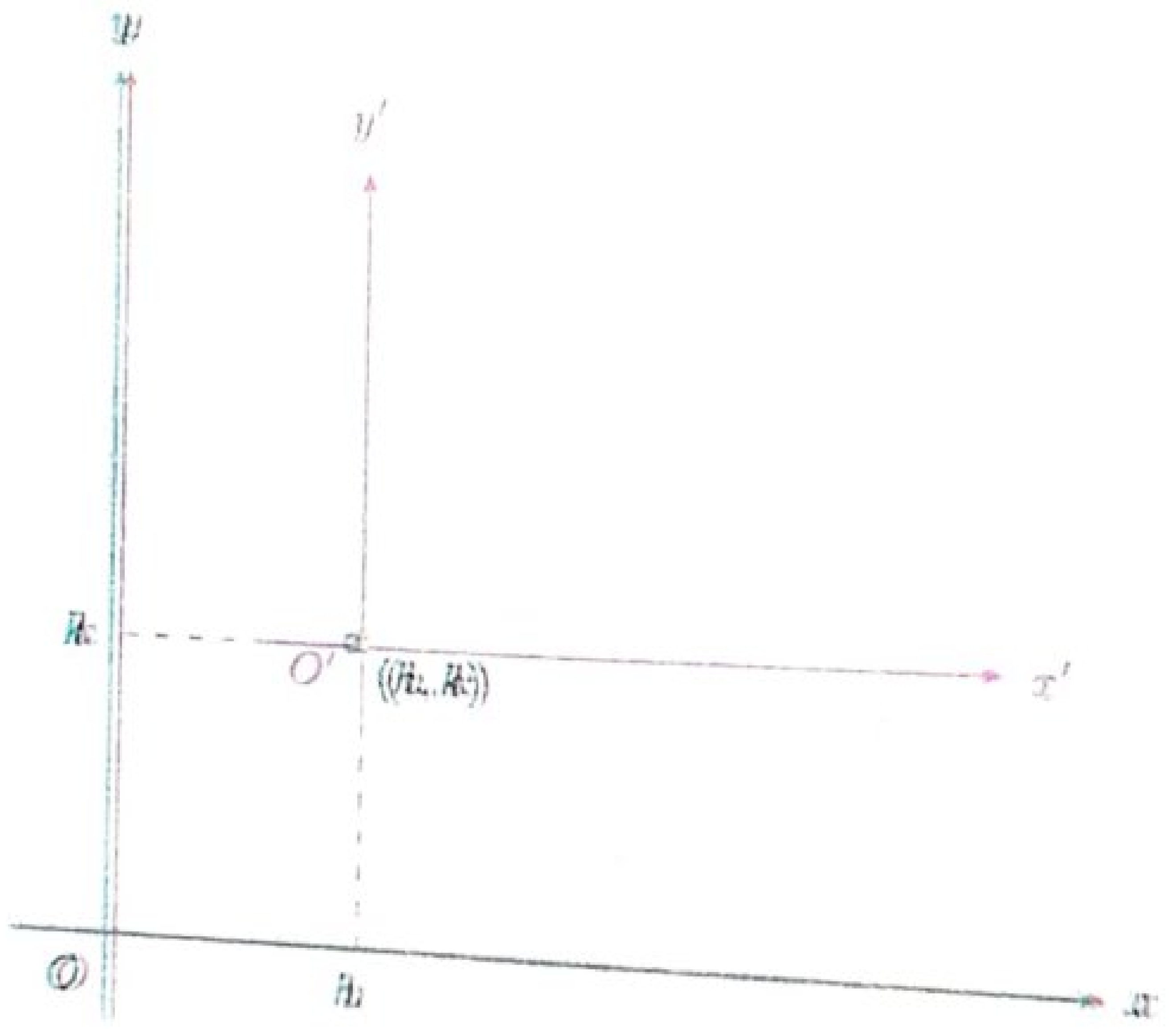
1. $y^2 = 8x$
2. $y^2 = -6x$
3. $x + y^2 = 0$
4. $y = x^2$
5. $y = -x^2$
6. $y = 3x^2$
7. $y = -3x^2$
8. $x^2 + 6y = 0$

In Exercise 9 - 16, find an equation for the parabola satisfying the given conditions.

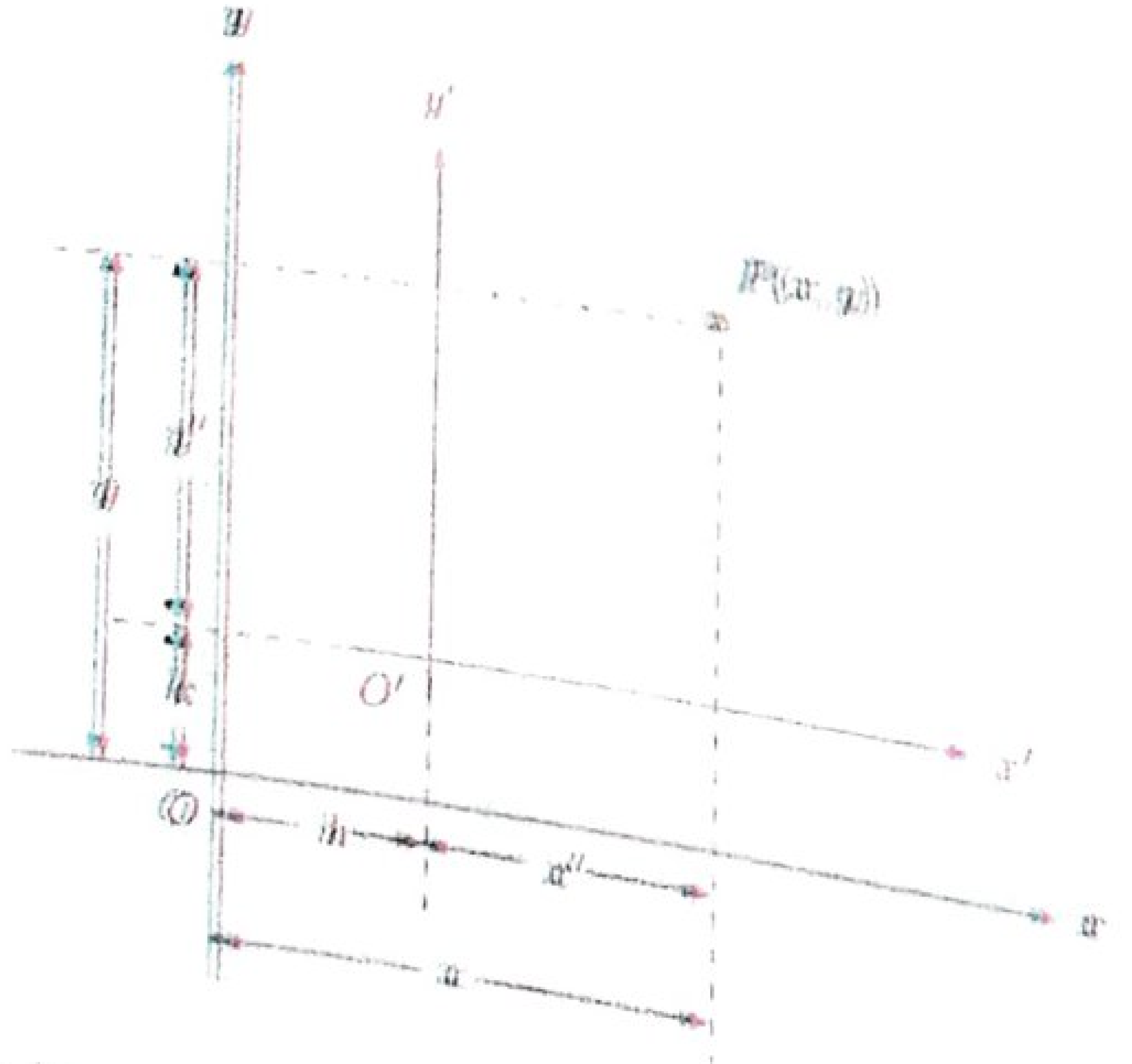
9. Vertex $(0, 0)$, focus $(3, 0)$.
10. Vertex $(0, 0)$, focus $(0, -4)$.
11. Vertex $(0, 0)$, directrix $x = 7$.
12. Vertex $(0, 0)$, directrix $y = \frac{1}{2}$.
13. Focus $(0, -3)$, directrix $y = 3$.
14. Focus $(6, 0)$, directrix $x = -6$.
15. Vertex $(0, 0)$; symmetric about the x-axis; passes through $(2, 2)$.
16. Vertex $(0, 0)$; symmetric about the y-axis; passes through $(-1, 3)$.

Translation of Axes

If a parabola with vertex at the origin and axis of symmetry along a coordinate axis is shifted horizontally h units and then vertically k units, then we get the parabola with vertex at (h, k) and axis of symmetry parallel to a coordinate axis.



In the figure, a new $x'y'$-coordinate system appears with the origin O' at the point (h, k).
A point P in the plane has both (x, y) coordinates and (x', y') coordinates.



In the figure, their coordinates are related by $x' = x - h, y' = y - k$ or equivalently, $x = x' + h, \quad y = y' + k$.
These are called **translation equations**.

As an example, if a new origin is $(h, k) = (4, -1)$ and the xy-coordinates of the point P are $(2, 5)$, then the $x'y'$-coordinates of P are

$$x' = x - h = 2 - 4 = -2 \text{ and}$$
$$y' = y - k = 5 - (-1) = 6\ .$$

Therefore the new coordinates of the point is $(-2, 6)$.

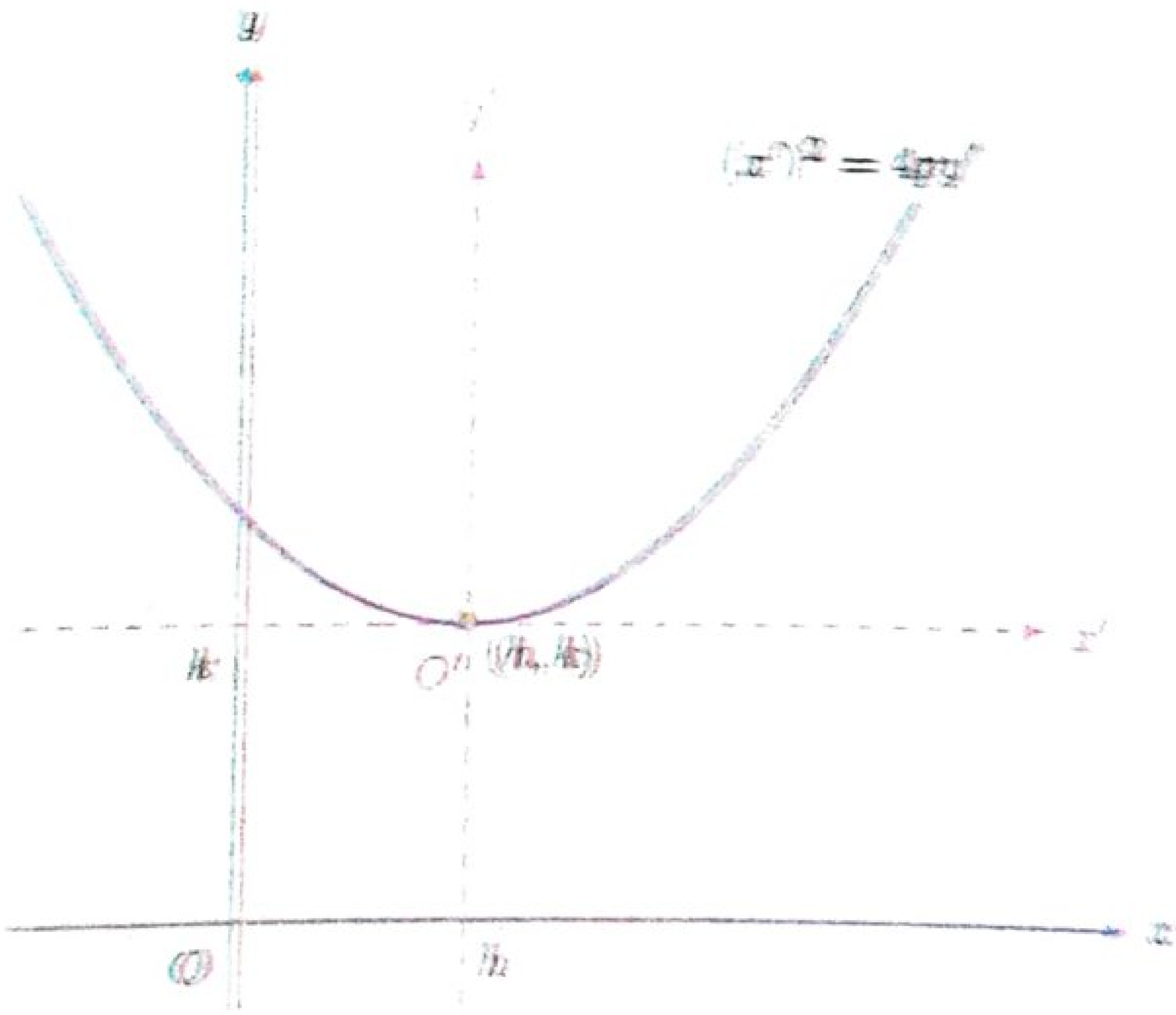
Translated Parabolas

Now we can consider a parabola with vertex V at (h, k) in any xy-coordinate system and axis of symmetry parallel to the y-axis. If we translate the axis, the vertex V will be at the origin of an $x'y'$-coordinate system.

Parabola with Vertex (h, k) and Axis parallel to y-axis

In $x'y'$-coordinates, the equation of the parabola will be

$$(x')^2 = 4py' \text{ or } (x')^2 = -4py' \text{ (See the figure).}$$



The corresponding equation in xy-coordinates will be

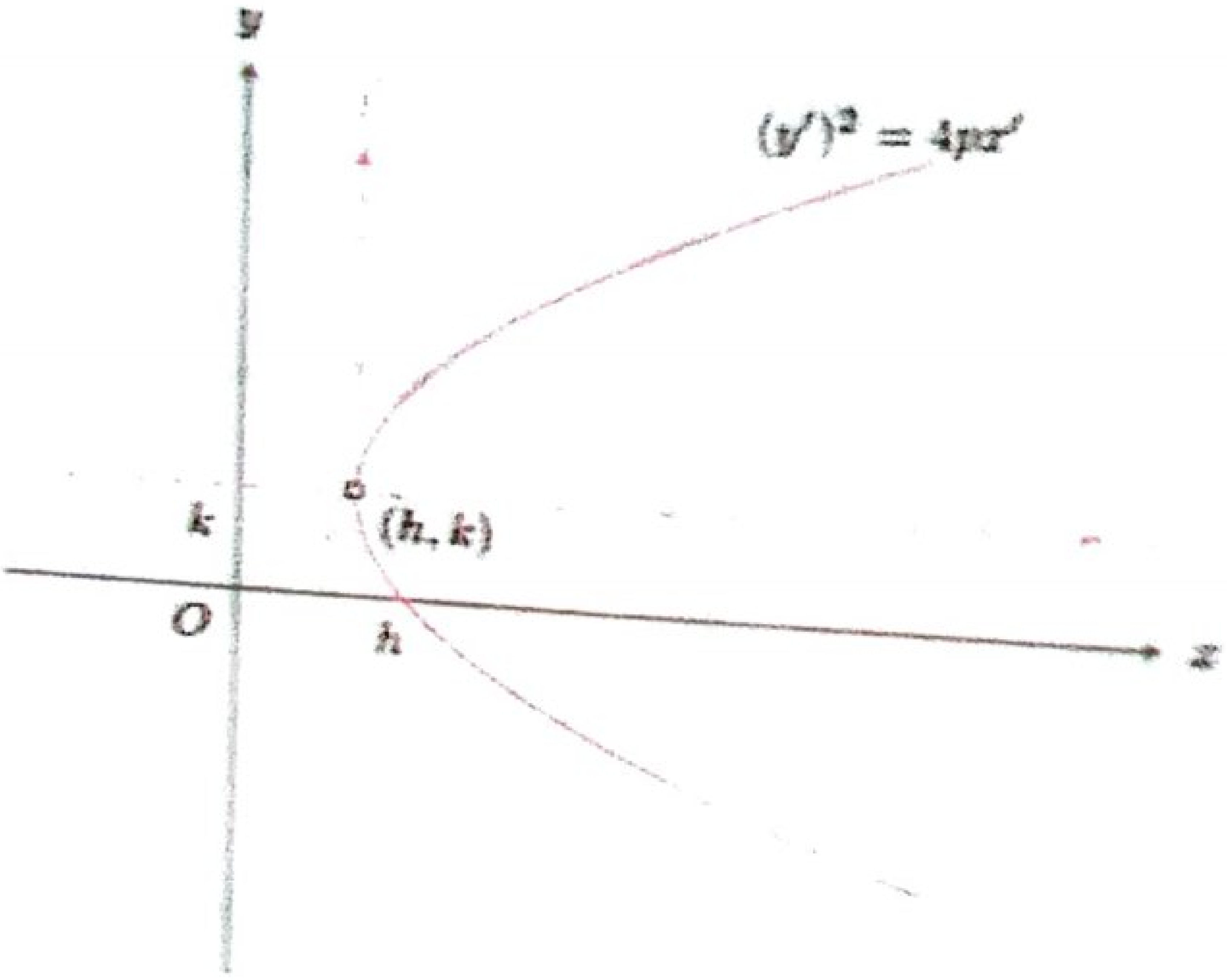
$$(x - h)^2 = \pm 4p(y - k).$$

Here the $(+)$ sign occurs if the parabola opens in the positive y direction and the $(-)$ sign occurs if the parabola opens in the negative y direction.

Parabola with Vertex (h, k) and Axis parallel to x-axis

The equation of the parabola with vertex (h, k) and axis parallel to x-axis is of the form

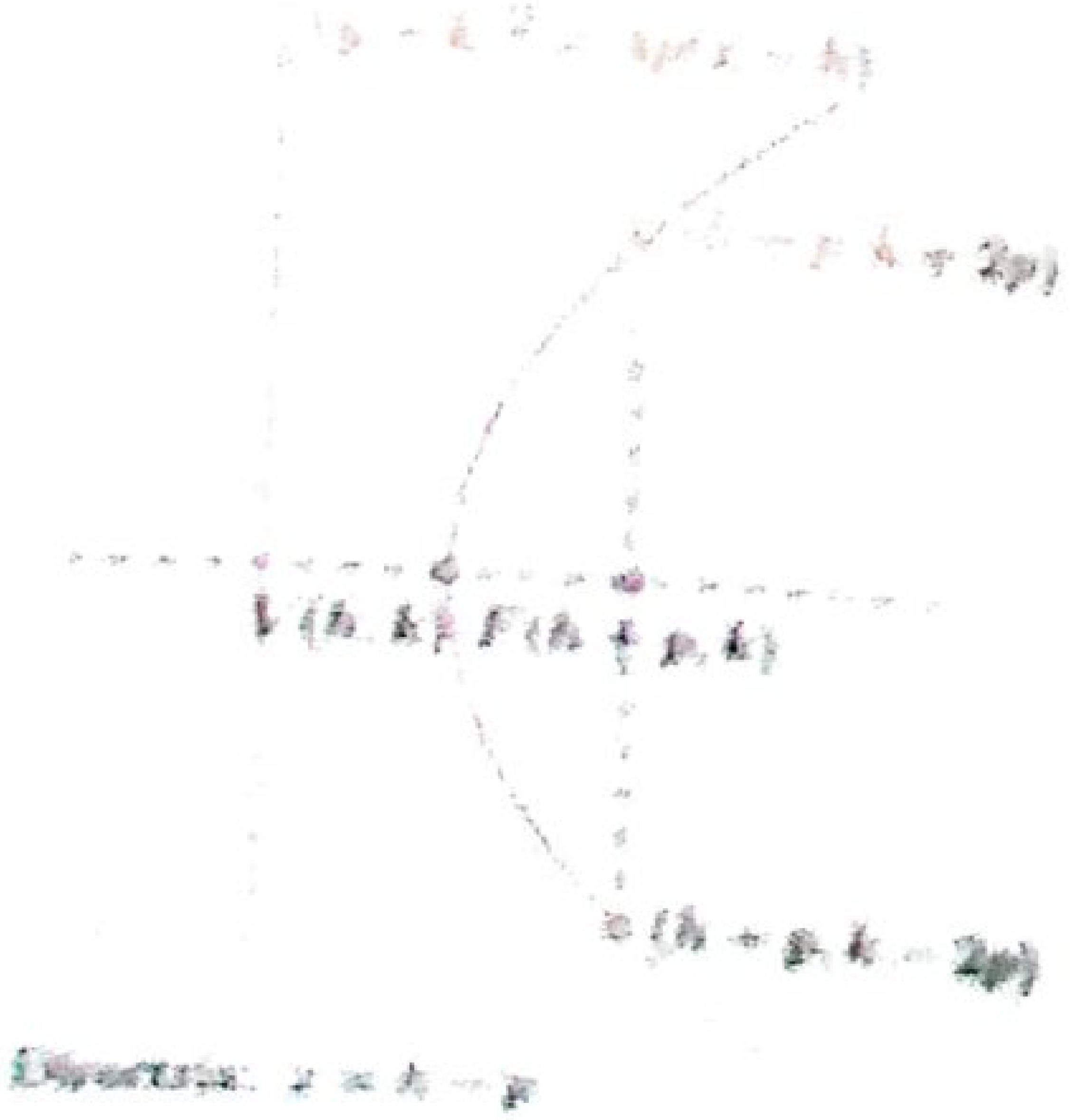
$$(y-k)^2 = \pm 4p(x-h).$$



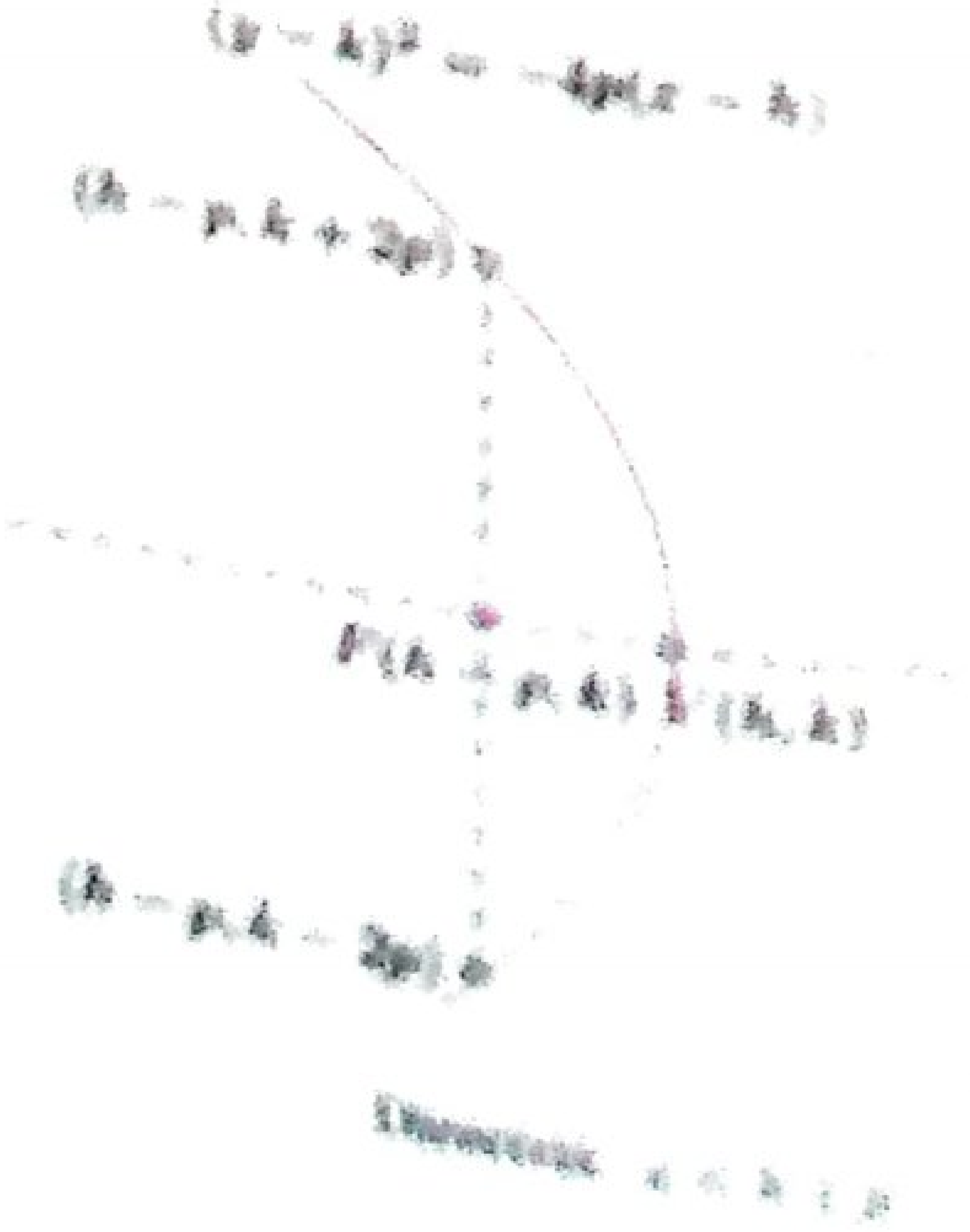
y
$(y')^2 = 4px'$
k
(h, k)
O
h
x

Here the $(+)$ sign occurs if the parabola opens in the positive x direction and the $(-)$ sign occurs if the parabola opens in the negative x direction.

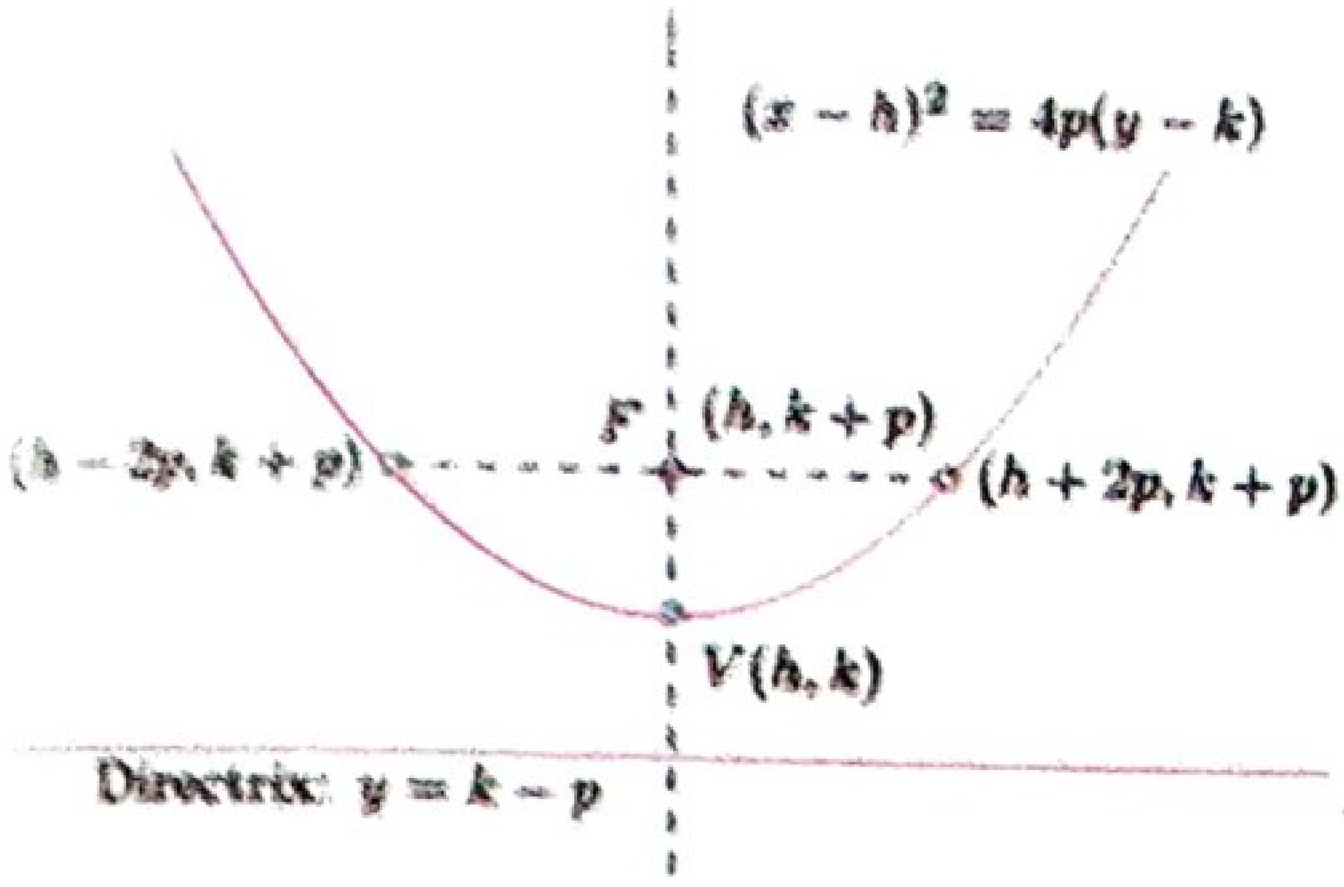
The four possible such orientations whose vertices not at the origin are shown in the following figures. These are called the standard positions of a parabola, and the resulting equations are called the **standard forms of equation of a parabola**.

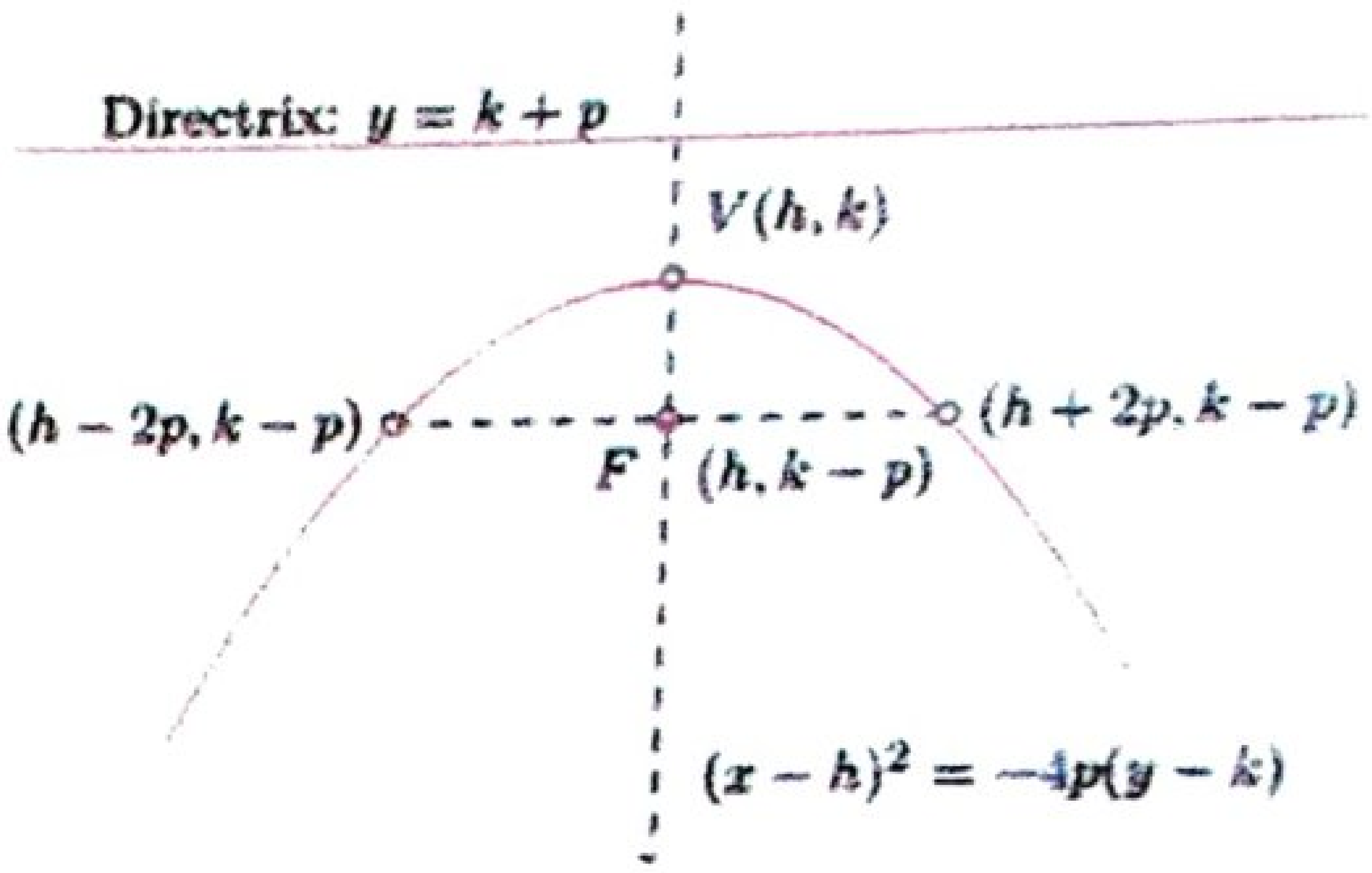


$(y-k)^2 = 4p(x-h)$
$(h+p, k+2p)$
$V(h, k)$
$F(h+p, k)$
$(h+p, k-2p)$
Directrix: $x = h - p$



$(y-k)^2 = -4p(x-h)$
$(h-p, k+2p)$
$F(h-p, k)$
$V(h, k)$
$(h-p, k-2p)$
Directrix: $x = h + p$


$(x-h)^2 = 4p(y-k)$
F
$(h, k+p)$
$(h-2p, k+p)$
$(h+2p, k+p)$
$V(h,k)$
Directrix: $y = k - p$


Directrix: $y = k + p$
$V(h,k)$
$(h-2p, k-p)$
$(h+2p, k-p)$
F
$(h, k-p)$
$(x-h)^2 = -4p(y-k)$

Equations of a Parabola:

Vertex at (h,k), Parallel to Coordinate Axis, $p > 0$

Equation	Vertex	Focus	Directrix	Axis of Symmetry	Description
$(y-k)^2 = 4p(x-h)$	(h,k)	$(h+p,k)$	$x = h-p$	horizontal axis, $y = k$	opens to the right
$(y-k)^2 = -4p(x-h)$	(h,k)	$(h-p,k)$	$x = h+p$	horizontal axis, $y = k$	opens to the left
$(x-h)^2 = 4p(y-k)$	(h,k)	$(h,k+p)$	$y = k-p$	vertical axis, $x = h$	opens up
$(x-h)^2 = -4p(y-k)$	(h,k)	$(h,k-p)$	$y = k+p$	vertical axis, $x = h$	opens down

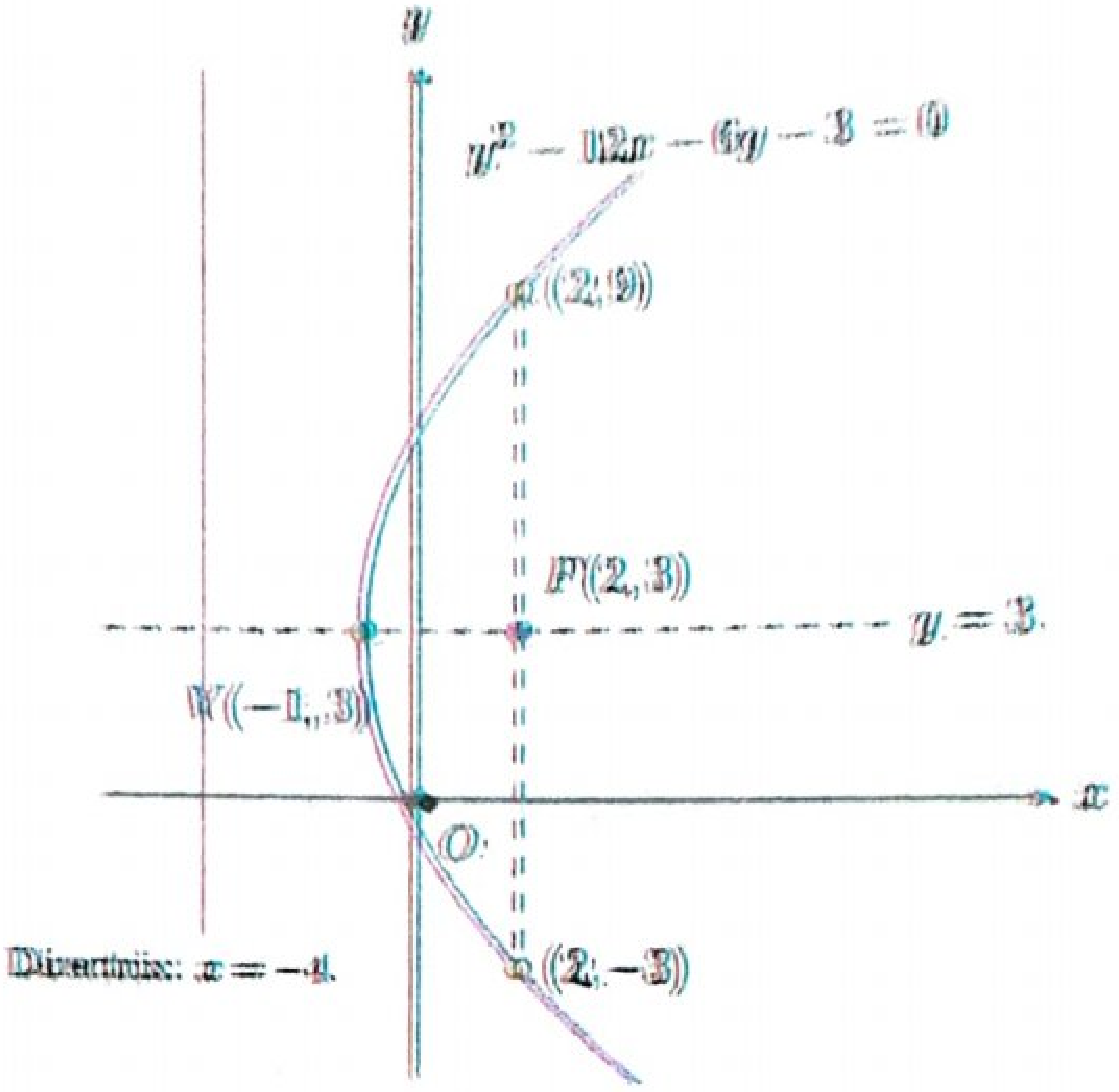
Example 8.

Write the given equation $y^2 - 12x - 6y - 3 = 0$ in standard form. Sketch its graph showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$\begin{aligned} y^2 - 12x - 6y - 3 &= 0 \\ y^2 - 6y &= 12x + 3 \\ y^2 - 6y + 3^2 &= 12x + 3 + 9 \\ (y-3)^2 &= 12x + 12 \\ (y-3)^2 &= 12(x+1) \end{aligned}$$

Comparing with $(y-k)^2 = 4p(x-h)$, we have $h = -1$, $k = 3$ and $p = 3$. Therefore, the vertex $V = (-1,3)$, the focus $F = (2,3)$ and the directrix is $x = -4$.


y
$y^2 - 12x - 6y - 3 = 0$
(2, 9)
F(2, 3)
y = 3
V(−1, 3)
x
O
Directrix: x = −4
(2, −3)

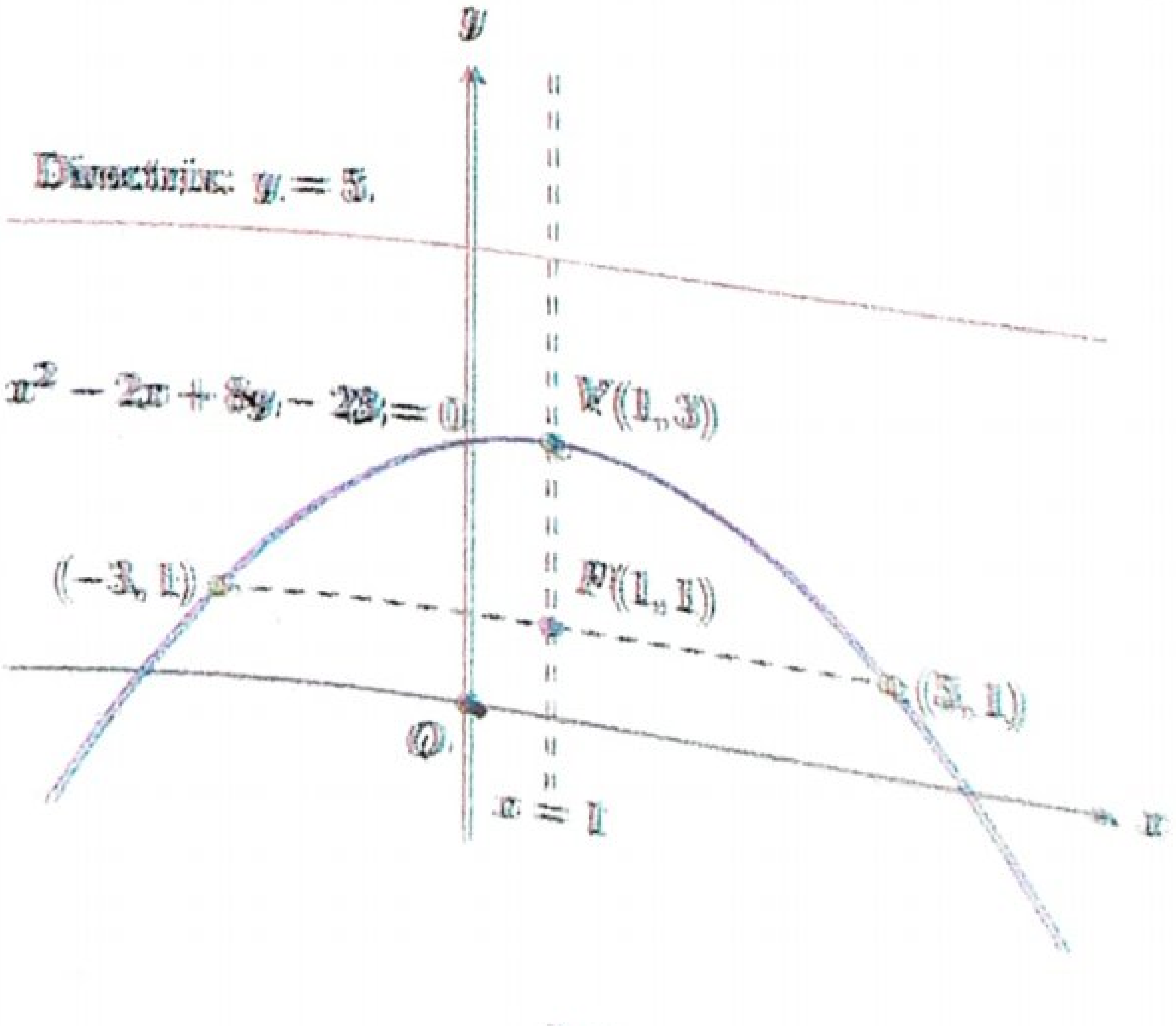
Example 9.

Write the given parabola equation $x^2 - 2x + 8y - 23 = 0$ in standard form. Sketch its graph showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$x^2 - 2x + 8y - 23 = 0$$
$$x^2 - 2x = -8y + 23$$
$$x^2 - 2x + 1^2 = -8y + 23 + 1$$
$$(x-1)^2 = -8y + 24$$
$$(x-1)^2 = -8(y-3)$$

Comparing with $(x-h)^2 = -4p(y-k)$, we have $h = 1$, $k = 3$ and $p = 2$. Therefore, the vertex $V = (1, 3)$, the focus $F = (1, 1)$ and the directrix is $y = 5$.


y
Directrix: y = 5
$x^2 - 2x + 8y - 23 = 0$
V(1, 3)
(−3, 1)
F(1, 1)
(5, 1)
O
x = 1
x

Example 10.

Find the standard form of the equation of the parabola with vertex $(1, 2)$ and focus at $(4, 2)$.

Solution

Since the focus and vertex are on a horizontal line and the focus is to the right of the vertex, the parabola opens to the right.
The vertex $V = (1, 2)$, the focus $F = (4, 2) = (1 + 3, 2)$.
Therefore $h = 1$, $k = 2$ and $p = 3$.
The equation of the parabola is

$$\begin{aligned}(y-k)^2 &= 4p(x-h)\\(y-2)^2 &= 4\times 3(x-1)\\(y-2)^2 &= 12(x-1).\end{aligned}$$

Example 11.

Find the general form of the equation of the parabola with vertex $(2, 1)$ and focus $(2, 4)$.

Solution

Since the focus and vertex are on a vertical line and the focus is above the vertex, the parabola opens up.
Then $V = (h, k) = (2, 1)$, $F = (2, 4) = (2, 1 + 3)$ and $p = 3$.
The equation of the parabola is

$$\begin{aligned}(x-h)^2 &= 4p(y-k)\\(x-2)^2 &= 4\times 3(y-1)\\x^2-4x+16 &= 12y\\x^2-4x-12y+16 &= 0.\end{aligned}$$

The general form of the parabola is $x^2 - 4x - 12y + 16 = 0$.

Exercise 6.3

In Exercise 1 - 6, (a) sketch the parabola, (b) show the focus, vertex, directrix and end points of latus rectum:

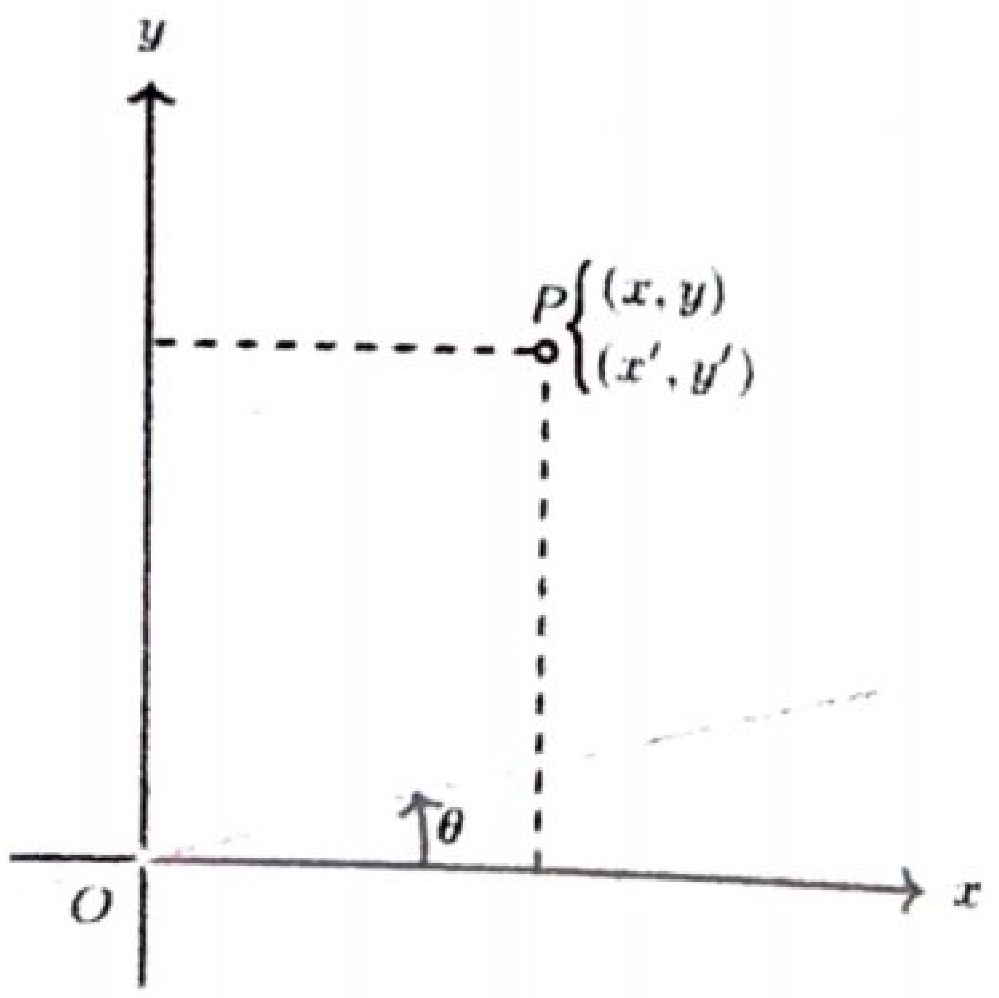
1. $x^2 + 4x + 4y = 0$
2. $(x-1)^2 + 8(y+1) = 0$
3. $(x+5) + (y-1)^2 = 0$
4. $y^2 + 6y + 8x + 25 = 0$
5. $x^2 + 4x + 6y - 2 = 0$
6. $y^2 + x + y = 0$

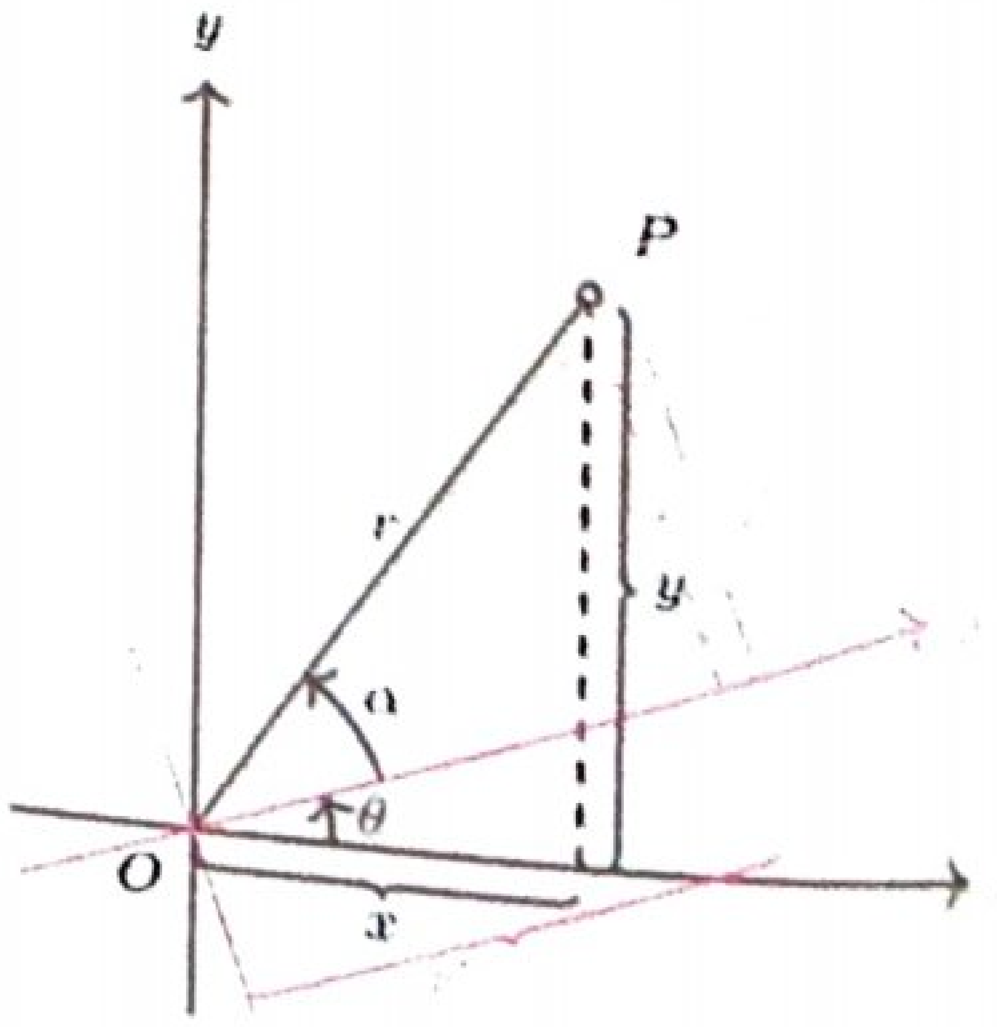
In Exercise 7 - 14, find an equation for the parabola satisfying the given conditions.

7. Vertex $(4,-5)$, focus $(1,-5)$.
8. Vertex $(5,1)$, focus $(5,3)$.
9. Vertex $(1,1)$, directrix $y=-2$.
10. Vertex $(2,-1)$, directrix $x=-1$.
11. Focus $(-1,4)$, directrix $x=5$.
12. Focus $(6,0)$, directrix $x=-6$.
13. Axis $x=0$ passes through $(2,-1)$ and $(-4,5)$.
14. Axis $y=0$ passes through $(3,2)$ and $(2,-3)$.

6.4 Rotation of Axes

In a rotation of axes, the origin remains fixed while the x-axis and the y-axis are rotated through an angle θ to a new position; the new positions of the x and y axes are denoted by x' and y', respectively as shown in figure.





Here, the point P has the coordinates (x,y) relative to the xy-plane. We can express x and y in terms of x', y' and θ.

As in figure, r denotes the distance from the origin to the point P, and α denotes the angle between the positive x'-axis and the ray from O through P. Then, by using definitions of sine and cosine, we have

$$x' = r\cos\alpha, \quad y' = r\sin\alpha,$$

$$x = r\cos(\theta+\alpha), \quad y = r\sin(\theta+\alpha).$$

Method 2

$$x^2 + y^2 - 2x + 10y + 23 = 0$$

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0,$

we get $2g = -2, \quad 2f = 10, \quad c = 23,$

and so $g = -1, \quad f = 5, \quad c = 23.$

Therefore, the center $= (-g, -f) = (1, -5)$ and

$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 25 - 23} = \sqrt{3}\ .$$

(b) Method 1

$$\begin{aligned}
2x^2 + 2y^2 - 24x + 47 &= 0\\
x^2 + y^2 - 12x + \frac{47}{2} &= 0\\
x^2 - 12x + y^2 &= -\frac{47}{2}\\
x^2 - 12x + 6^2 + y^2 &= 36 - \frac{47}{2}\\
(x-6)^2 + y^2 &= \frac{25}{2}\\
(x-6)^2 + y^2 &= (\frac{5}{\sqrt{2}})^2
\end{aligned}$$

The center $= (6, 0)$ and radius $= \dfrac{5}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$.

Method 2

$$\begin{aligned}
2x^2 + 2y^2 - 24x + 47 &= 0\\
x^2 + y^2 - 12x + \frac{47}{2} &= 0
\end{aligned}$$

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0,$

we get $2g = -12, \quad 2f = 0, \quad c = \dfrac{47}{2},$

and so $g = -6, \quad f = 0.$

Therefore, the center $= (-g, -f) = (6, 0)$ and

$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{36 + 0 - \frac{47}{2}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}.$$

Example 5.

Describe the graphs of the following equations.

(a) $(x-1)^2+(y+4)^2=-9$

(b) $(x-1)^2+(y+4)^2=0$

(c) $x^2+y^2-4x+2y-44=0$

Solution

(a) $(x-1)^2+(y+4)^2=-9$

We see that the right hand side is less than zero, so the equation has no graph.

(b) $(x-1)^2+(y+4)^2=0$

The graph is a single point $(1,-4)$.

(c) $x^2+y^2-4x+2y-44=0$

Comparing with $\quad x^2+y^2+2gx+2fy+c=0,$

we get $\quad 2g=-4, \quad 2f=2, \quad c=-44,$

and so $\quad g=-2, \quad f=1, \quad c=-44.$

Since $\quad g^2+f^2-c=4+1+44=49>0,$

the graph is a circle with center at $(2,-1)$ and radius is 7.

Exercise 6.1

1. Find the center and radius of each circle:

   (a) $x^2+y^2=16$

   (b) $(x-1)^2+(y-4)^2=25$

   (c) $(x-3)^2+(y+5)^2=49$

   (d) $(x+1)^2+y^2=1$

   (e) $2x^2+2y^2-12x+8y-24=0$

   (f) $x^2+y^2+x+y-\frac{1}{2}=0$

2. Find the standard equation of the circle satisfying the given conditions.

   (a) center$(3,-2)$, radius$=4$.

   (b) center$(2,5)$, radius$=\sqrt{5}$.

   (c) center$(-4,1)$, radius$=3$.

   (d) center$(-4,8)$, circle is tangent to the x-axis.

   (e) center$(5,8)$, circle is tangent to the y-axis.

   (f) center$(-3,-2)$, circle passes through the origin.

   (g) center$(4,-5)$, circle passes through $(1,3)$.

   (h) diameter has end points $(6,1)$ and $(-2,3)$.

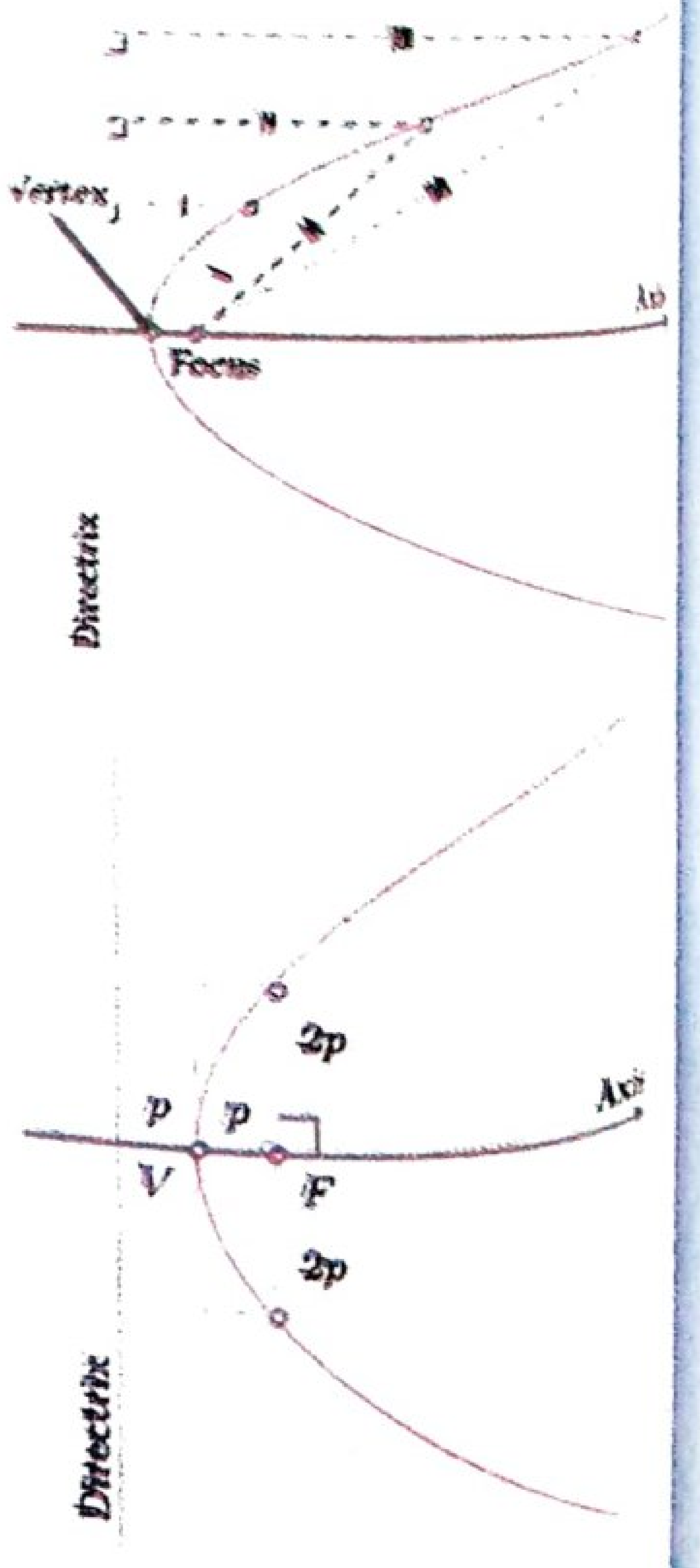
3. Determine whether each equation represents a circle, a point or no graph. If th equation represents a circle, find the center and radius.

(a) $x^2 + y^2 - 6x + 2y + 9 = 0$
(b) $x^2 + y^2 - 4x - 6y + 13 = 0$
(c) $x^2 + y^2 + 10y + 26 = 0$
(d) $2x^2 + 2y^2 - 12x + 8y - 24 = 0$
(e) $x^2 + y^2 - 10x - 2y + 29 = 0$
(f) $16x^2 + 16y^2 + 40x + 16y - 7 = 0$
(g) $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$
(h) $2x^2 + 2y^2 + 8x + 7 = 0$

.3 The Parabola

 parabola is the set of all points (x, y) in the plane that are equidistant from a fixed ne and a fixed point which is not on the fixed line.

n this section, we will discuss the properties of the arabolas. The fixed line is called the **directrix** f the parabola and the fixed point is the **focus.** he line through the focus and perpendicular to he directrix is called the **axis** of the parabola. The oint of intersection of the axis and the parabola is alled the **vertex.** A parabola is symmetric about he axis.



enote the distance between the focus F and the ertex V by p. The vertex is equidistant from the ocus and the directrix. Therefore, the distance etween focus and directrix is $2p$. As illustrated n the figure, the parabola passes through two f the corners of the box that extends from the ertex to the focus along the axis of symmetry and xtends $2p$ units above and $2p$ units below the axis f symmetry. The line segment passes through the ocus and is parallel to the directrix is called the **atus rectum**. The end points of the latus rectum e on the curve.

eneral Form of the Equation of a Parabola

rom the general form of the conics equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, we gin with the case where $B=0$. In this case, the term containing xy is not present nd it becomes

$$Ax^2+Cy^2+Dx+Ey+F=0.$$

urthermore, one of A and C equals 0 becomes a parabola. We have already discussed he procedure for identifying the graph of this kind of equation; we complete the squares f the quadratic expressions in x or y, or both. Then we have

$$Ax^2+Dx+Ey+F=0$$

r

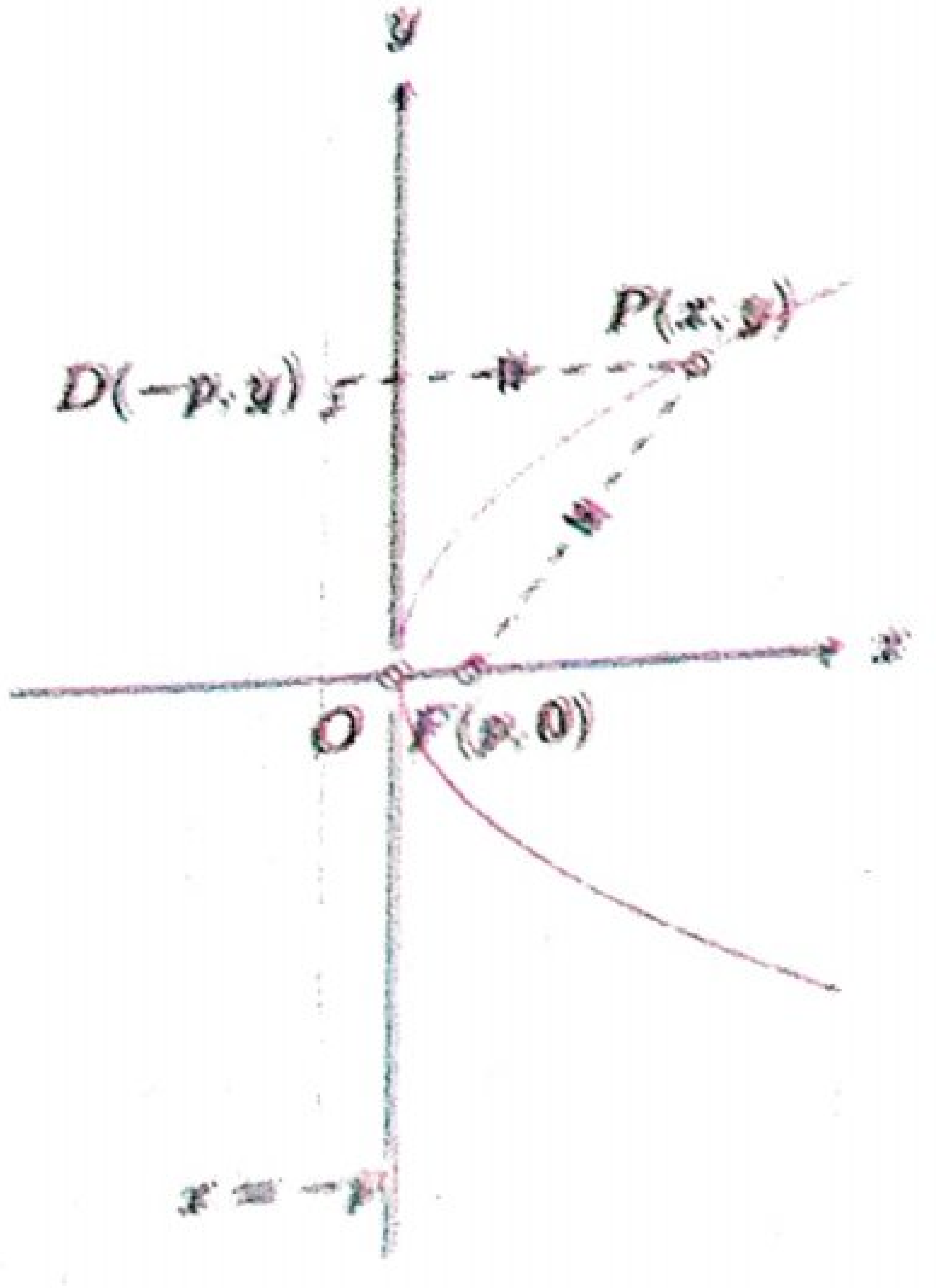
$$Cy^2+Dx+Ey+F=0.$$

bove two equations are called the **general forms of equation of a parabola** . e have learned that the graph of the quadratic function $y=ax^2+bx+c$, $a\neq 0$ is a arabola. If $a>0$, the graph opens up and otherwise, it opens down.

quations of a Parabola: Vertex at $(0,0)$

he equation of a parabola is simplest if the coordinate axes are positioned so that he vertex is at the origin and the axis is along the x-axis or y-axis.

e will derive the equation for the parabola. Let $P(x,y)$ be any point on the arabola. Since P is equidistant from the focus $F(p,0)$ and $D(-p,y)$ is the foot of the erpendicular from P to the directrix, where $p>0$, we have $PF=PD$.



By the distance formula,

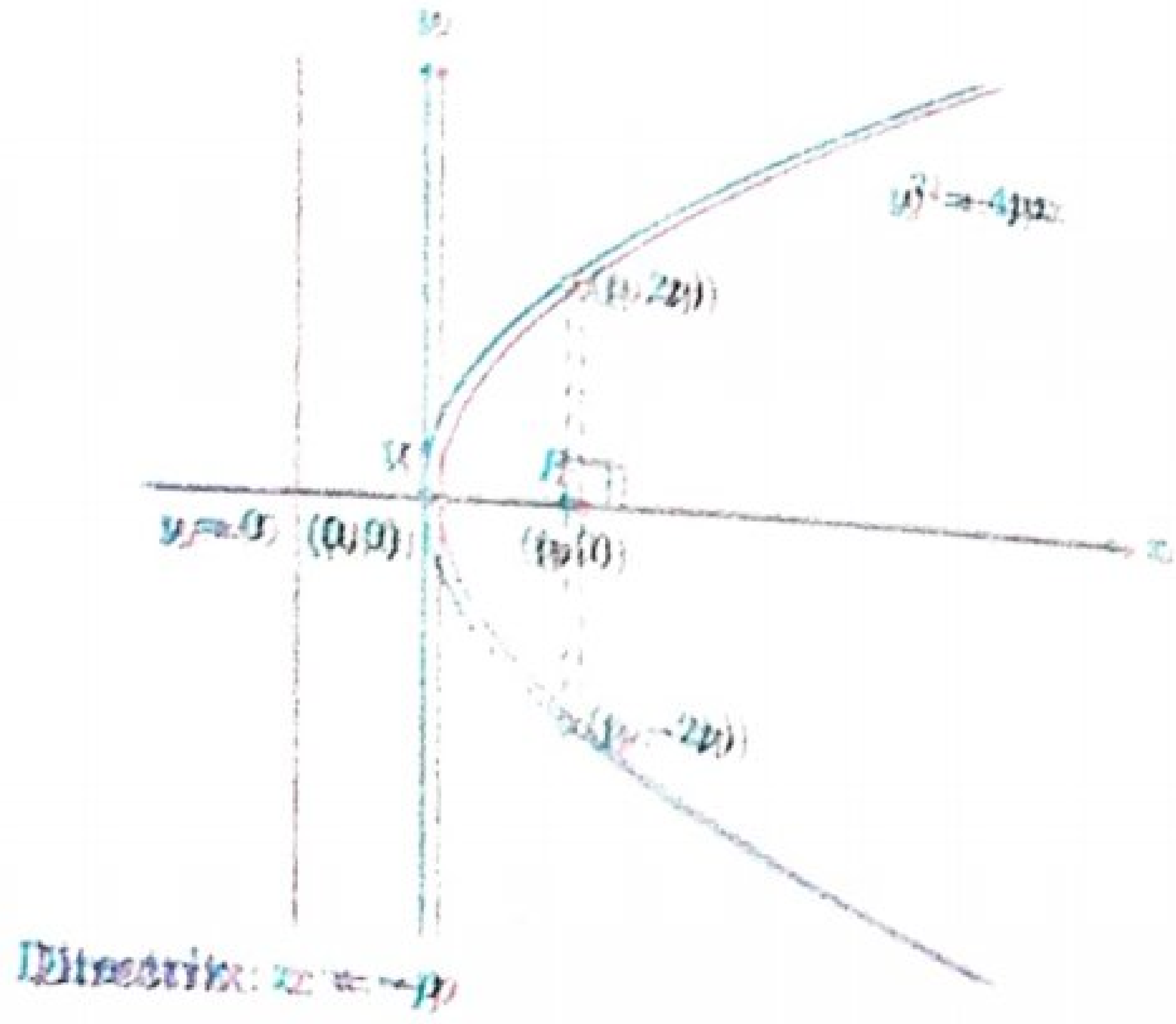
$$PF = \sqrt{(p-x)^2 + (0-y)^2},$$

$$PD = \sqrt{(-p-x)^2 + (y-y)^2}.$$

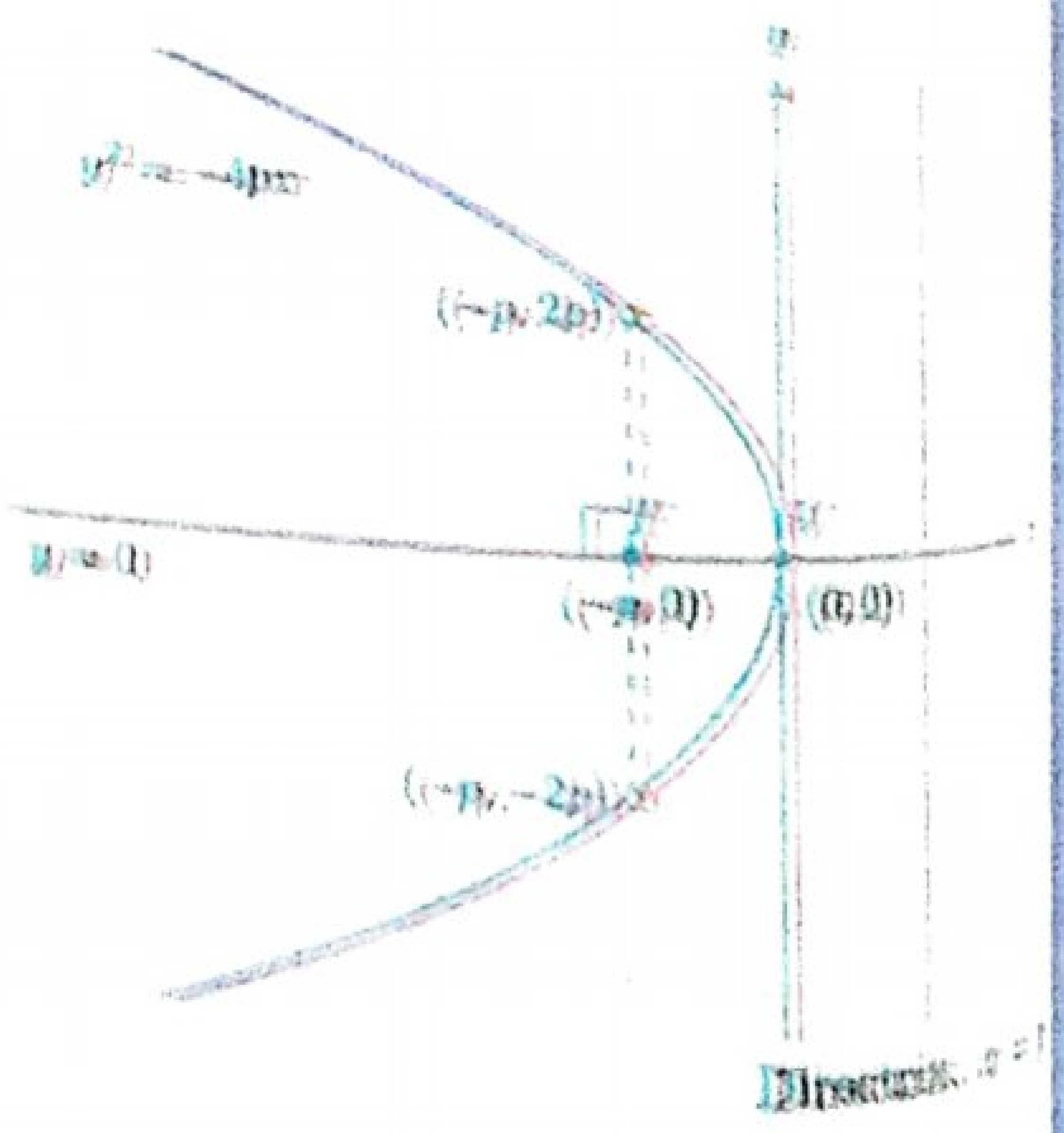
Since $PF = PD$, we have

$$\begin{aligned}\sqrt{(p-x)^2 + (0-y)^2} &= \sqrt{(-p-x)^2 + (y-y)^2}\\ p^2 - 2px + x^2 + y^2 &= p^2 + 2px + x^2\\ y^2 &= 4px.\end{aligned}$$

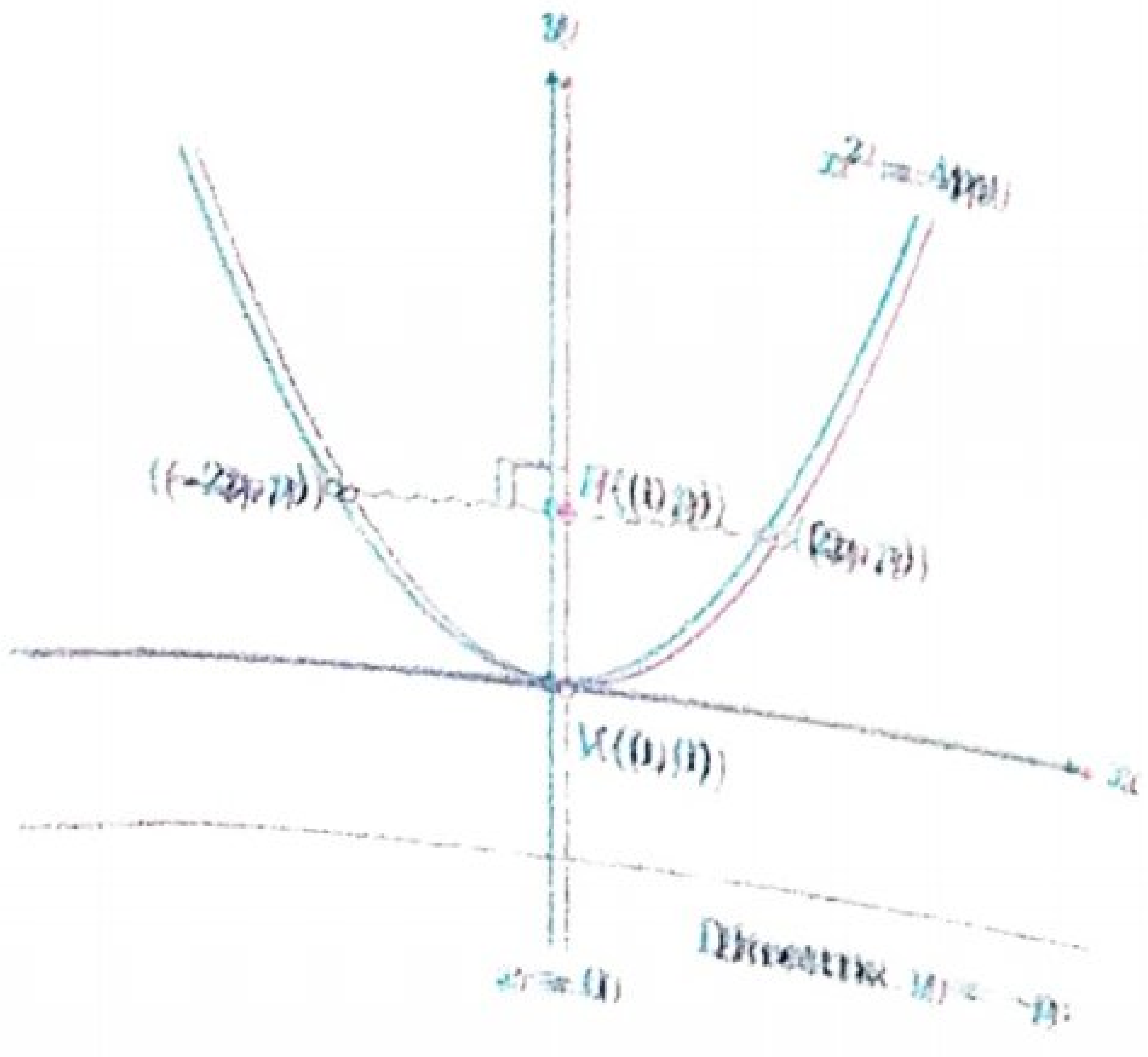
The four possible orientations are shown in the following figures. These are called th standard positions of a parabola, and the resulting equations are called the standar **forms of equation of a parabola.**



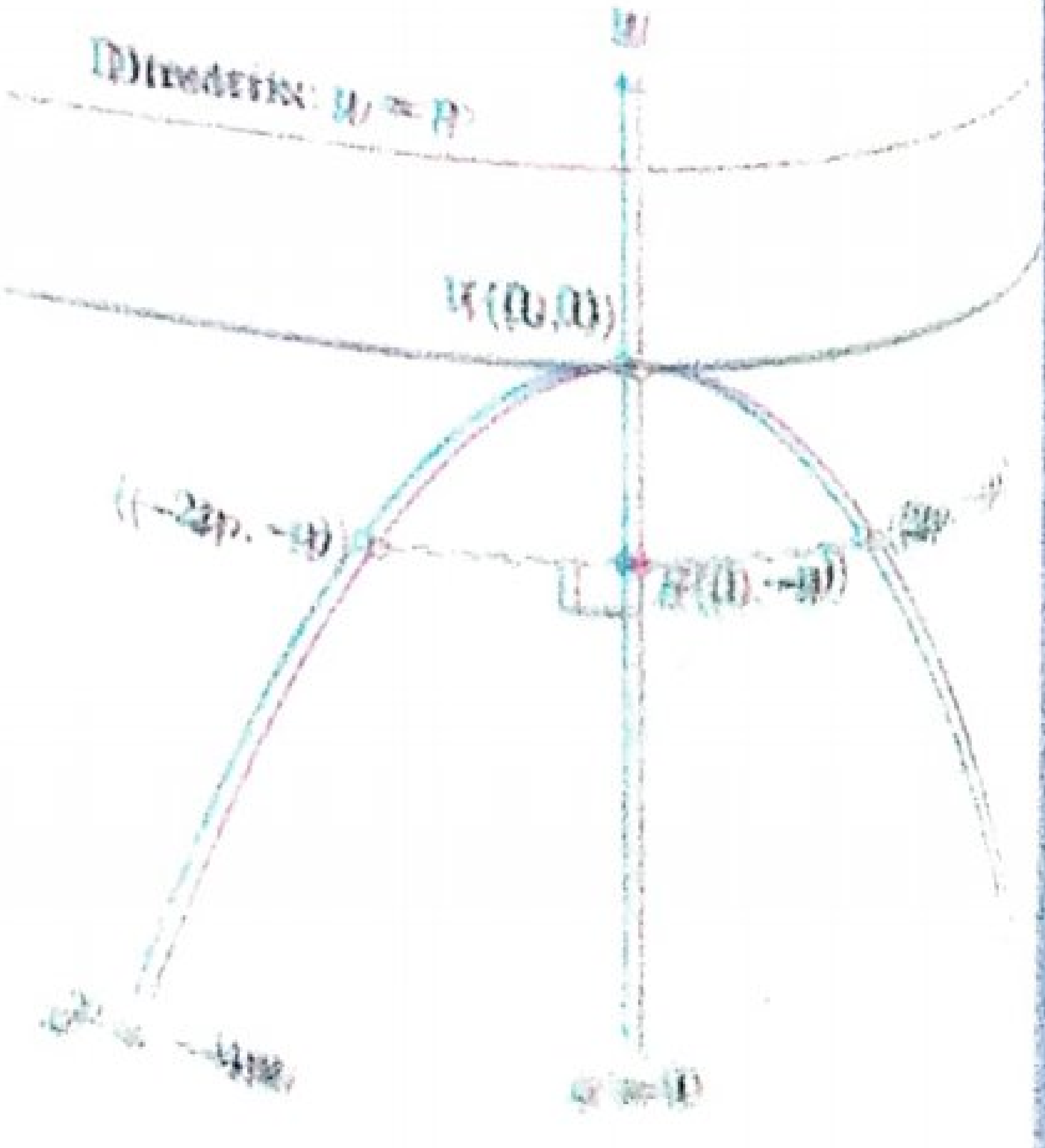
y
$y^2 = 4px$
(p, 2p)
V
F
y = 0
(0, 0)
(p, 0)
x
(p, −2p)
Directrix: x = −p



y
$y^2 = -4px$
(−p, 2p)
F
V
y = 0
(−p, 0)
(0, 0)
(−p, −2p)
Directrix: x =



y
$x^2 = 4py$
(−2p, p)
F(0, p)
(2p, p)
V(0, 0)
x
Directrix: y = −p
x = 0



y
Directrix: y = p
V(0, 0)
(−2p, −p)
(2p, −p)
F(0, −p)
$x^2 = -4py$
x = 0

Equations of a Parabola:

Vertex at $(0,0)$, Focus on Coordinate Axis, $p > 0$

Equation	Vertex	Focus	Directrix	Axis of Symmetry	Description
$y^2 = 4px$	$(0,0)$	$(p,0)$	$x = -p$	horizontal axis, $y = 0$	opens to the right
$y^2 = -4px$	$(0,0)$	$(-p,0)$	$x = p$	horizontal axis, $y = 0$	opens to the left
$x^2 = 4py$	$(0,0)$	$(0,p)$	$y = -p$	vertical axis, $x = 0$	opens up
$x^2 = -4py$	$(0,0)$	$(0,-p)$	$y = p$	vertical axis, $x = 0$	opens down

Parabolas can be graphed by using the following four steps :

- The axis of symmetry is along the x-axis if the equation has a y^2 term, and it is along the y-axis if the equation has a x^2 term.
- If the axis of symmetry is along the x-axis, then the parabola
  - opens to the right if the coefficient of x is positive.
  - opens to the left if the coefficient of x is negative.
- If the axis of symmetry is along the y-axis, then the parabola
  - opens up if the coefficient of y is positive.
  - opens down if the coefficient of y is negative.
- Determine the value of p from the standard equation and draw a box extending p units from the origin along the axis of symmetry in the direction in which the parabola opens and extending $2p$ units on each side of the axis of symmetry.

Using the above facts, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box.

Example 6.

Sketch the graphs of the parabolas, showing the vertex, focus, directrix and end points of latus rectum of each equation.

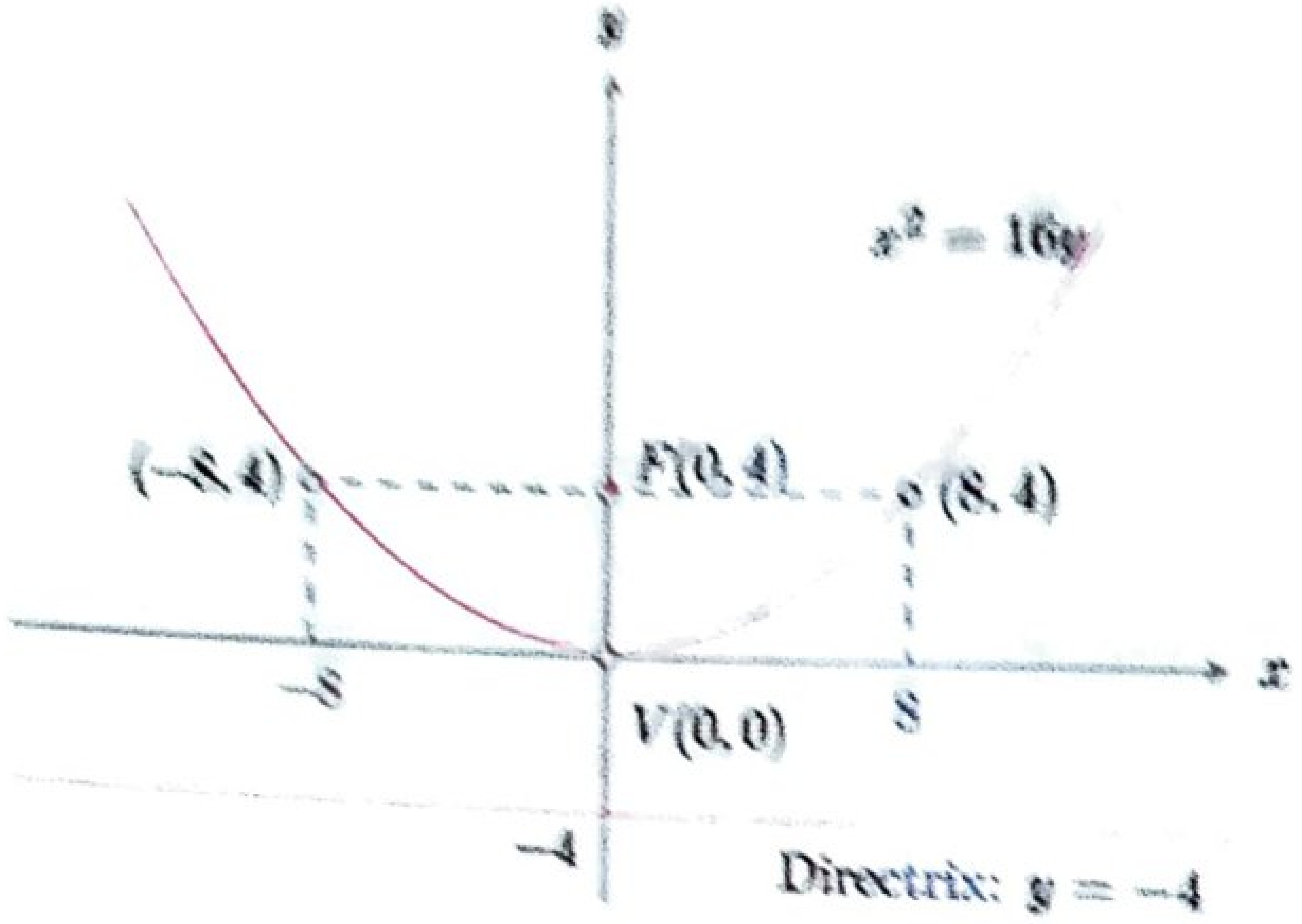
(a) $x^2 = 16y$ (b) $y^2 + 8x = 0$

Solution

(a) $x^2 = 16y$

Comparing with $x^2 = 4py$, we get $p = 4$.

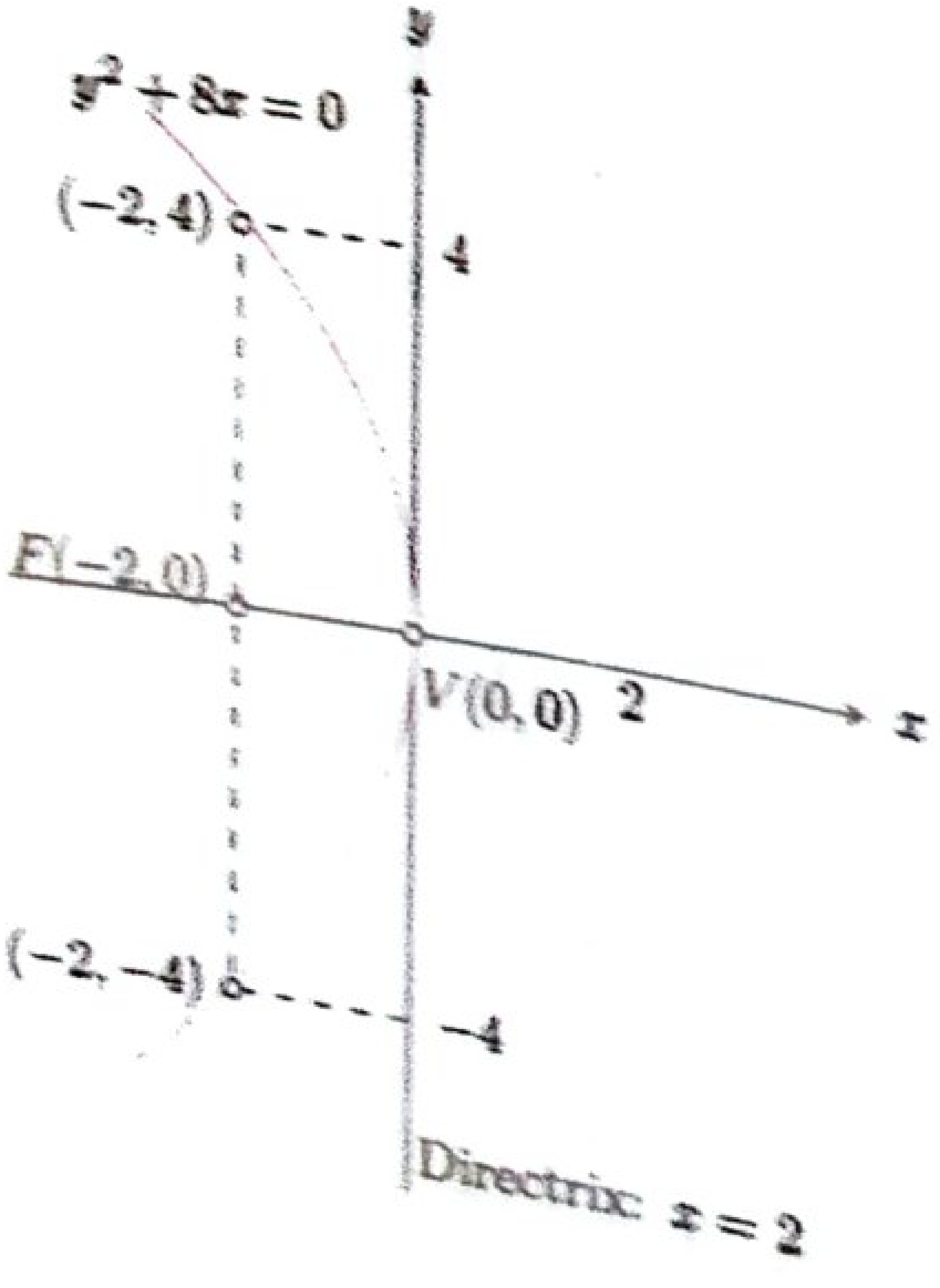
Therefore, the vertex $V = (0,0)$, the focus $F = (0,4)$, the directrix is $y = -4$ and end points of latus rectum are $(-8,4), (8,4)$.


y
$x^2 = 16y$
$(-8,4)$
$F(0,4)$
$(8,4)$
-8
$V(0,0)$
8
x
-4
Directrix: $y = -4$

(b) $y^2 + 8x = 0$

$y^2 = -8x$

Comparing with $y^2 = -4px$, we get $p = 2$.
Therefore, the vertex $V = (0,0)$, the focus $F(-2,0)$, the directrix is $x = 2$ and end points of latus rectum are $(-2,-4)$, $(-2,4)$.


y
$y^2 + 8x = 0$
$(-2,4)$
4
$F(-2,0)$
$V(0,0)$
2
x
$(-2,-4)$
-4
Directrix: $x = 2$

Example 7.

Find an equation of the parabola that is symmetric about the y-axis with its vertex at the origin, and passes through $(5, 2)$.

Solution
Since the parabola is symmetric about the y-axis, the vertex $V = (0, 0)$ and passes through $(5, 2)$, the equation of the form is

$$\begin{aligned} x^2 &= 4py \\ 5^2 &= 4p(2) \\ 4p &= \frac{25}{2}. \end{aligned}$$

Therefore, the equation of the parabola is

$$x^2 = \frac{25}{2}y.$$

Exercise 6.2

In Exercise 1 - 8, (a) sketch the parabola, (b) show the focus, vertex, directrix and end points of latus rectum.

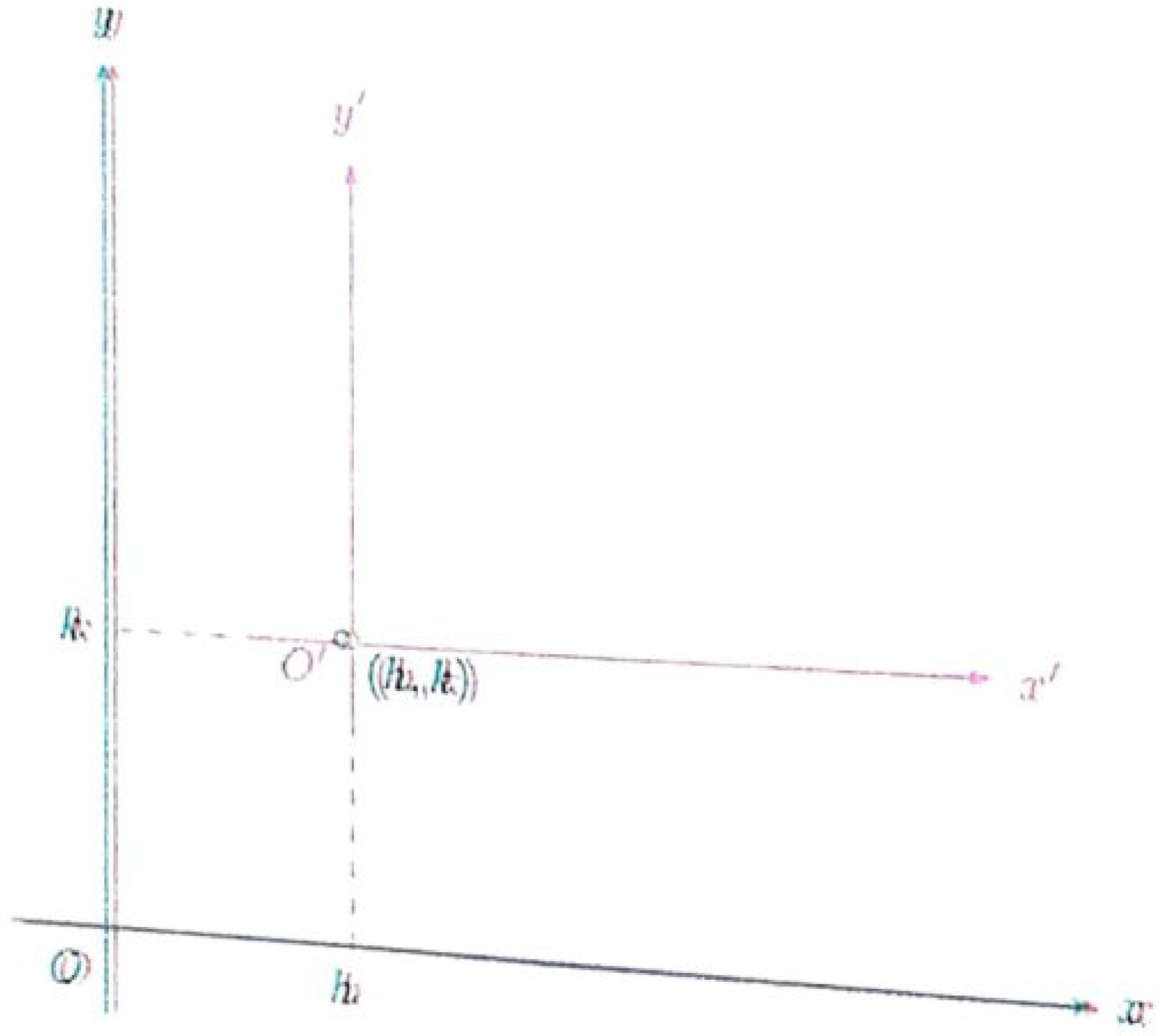
1. $y^2 = 8x$
2. $y^2 = -6x$
3. $x + y^2 = 0$
4. $y = x^2$
5. $y = -x^2$
6. $y = 3x^2$
7. $y = -3x^2$
8. $x^2 + 6y = 0$

In Exercise 9 - 16, find an equation for the parabola satisfying the given conditions.

9. Vertex $(0, 0)$, focus $(3, 0)$.
10. Vertex $(0, 0)$, focus $(0, -4)$.
11. Vertex $(0, 0)$, directrix $x = 7$.
12. Vertex $(0, 0)$, directrix $y = \frac{1}{2}$.
13. Focus $(0, -3)$, directrix $y = 3$.
14. Focus $(6, 0)$, directrix $x = -6$.
15. Vertex $(0, 0)$; symmetric about the x-axis; passes through $(2, 2)$.
16. Vertex $(0, 0)$; symmetric about the y-axis; passes through $(-1, 3)$.

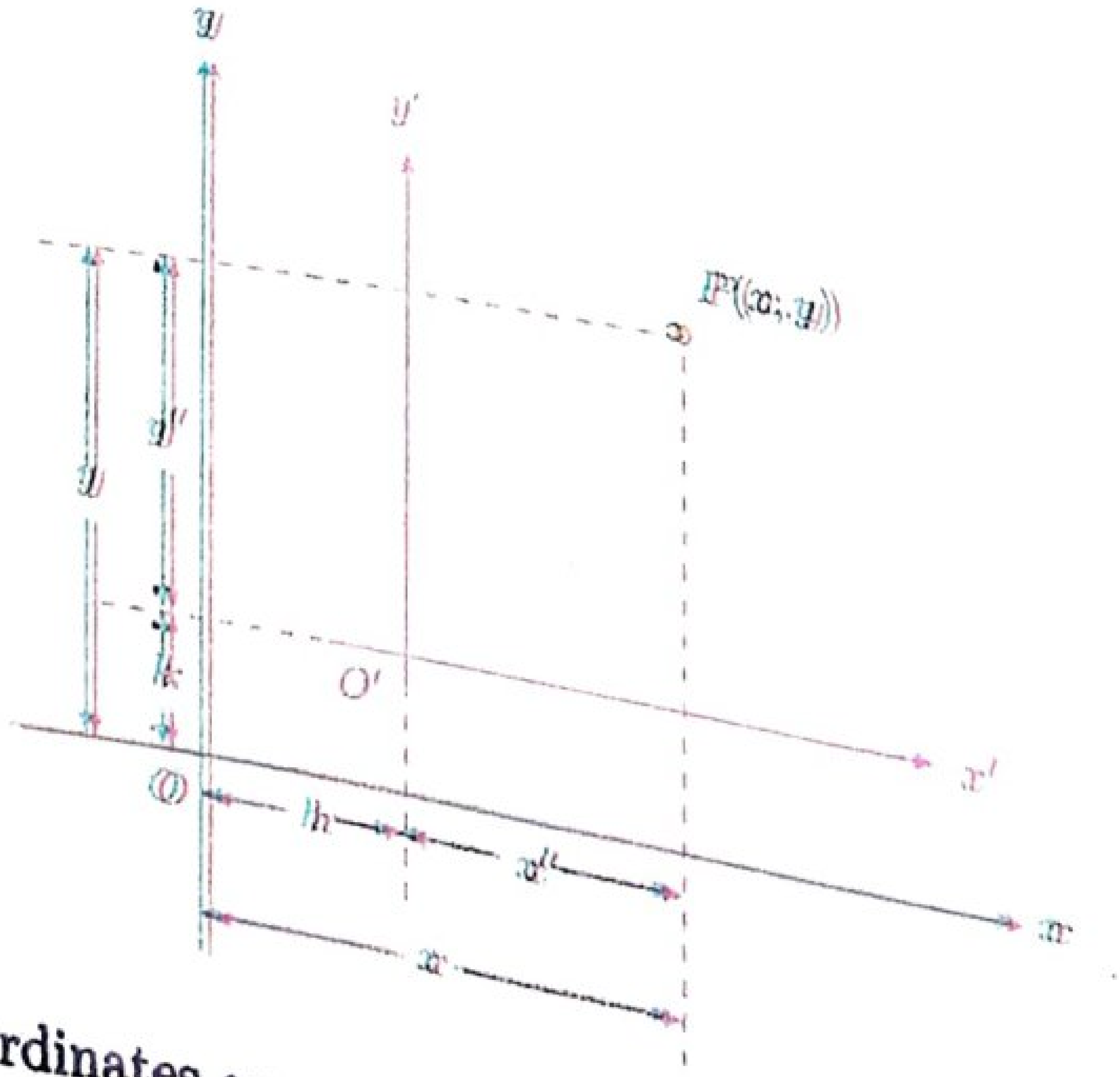
Translation of Axes

If a parabola with vertex at the origin and axis of symmetry along a coordinate axis is shifted horizontally h units and then vertically k units, then we get the parabola with vertex at (h, k) and axis of symmetry parallel to a coordinate axis.



In the figure, a new $x'y'$-coordinate system appears with the origin O' at the point (h, k).

A point P in the plane has both (x, y) coordinates and (x', y') coordinates.



In the figure, their coordinates are related by $x' = x - h, y' = y - k$ or equivalently,

$$x = x' + h, \quad y = y' + k.$$

These are called **translation equations.**

As an example, if a new origin is $(h,k) = (4,-1)$ and the xy-coordinates of the point P are $(2,5)$, then the $x'y'$-coordinates of P are

$x' = x - h = 2 - 4 = -2$ and

$y' = y - k = 5 - (-1) = 6$.

Therefore the new coordinates of the point is $(-2,6)$.

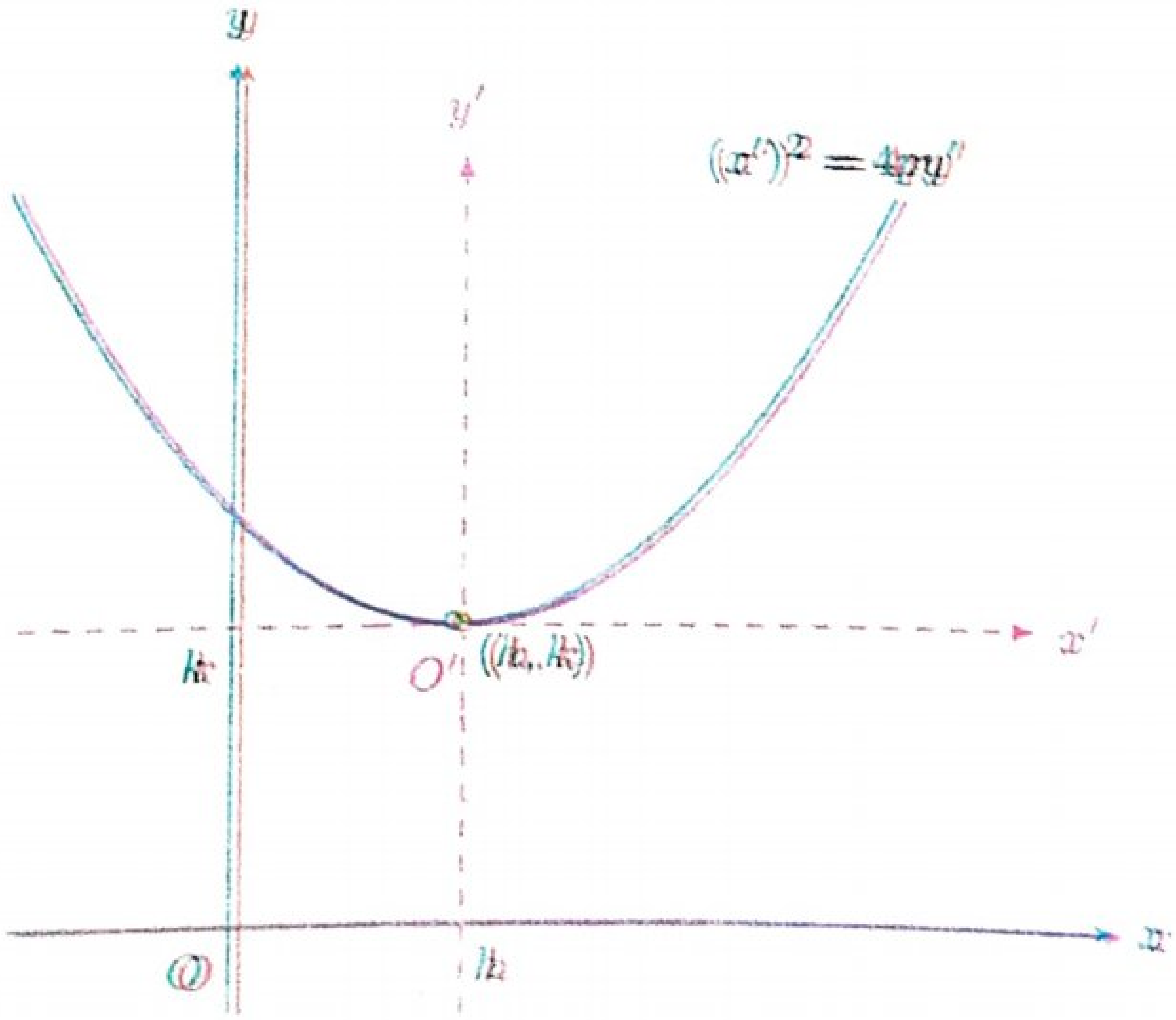
Translated Parabolas

Now we can consider a parabola with vertex V at (h,k) in any xy-coordinate system and axis of symmetry parallel to the y-axis. If we translate the axis, the vertex V will be at the origin of an $x'y'$-coordinate system.

Parabola with Vertex (h,k) and Axis parallel to y-axis

In $x'y'$-coordinates, the equation of the parabola will be

$$(x')^2 = 4py' \text{ or } (x')^2 = -4py' \text{ (See the figure).}$$



The corresponding equation in xy-coordinates will be

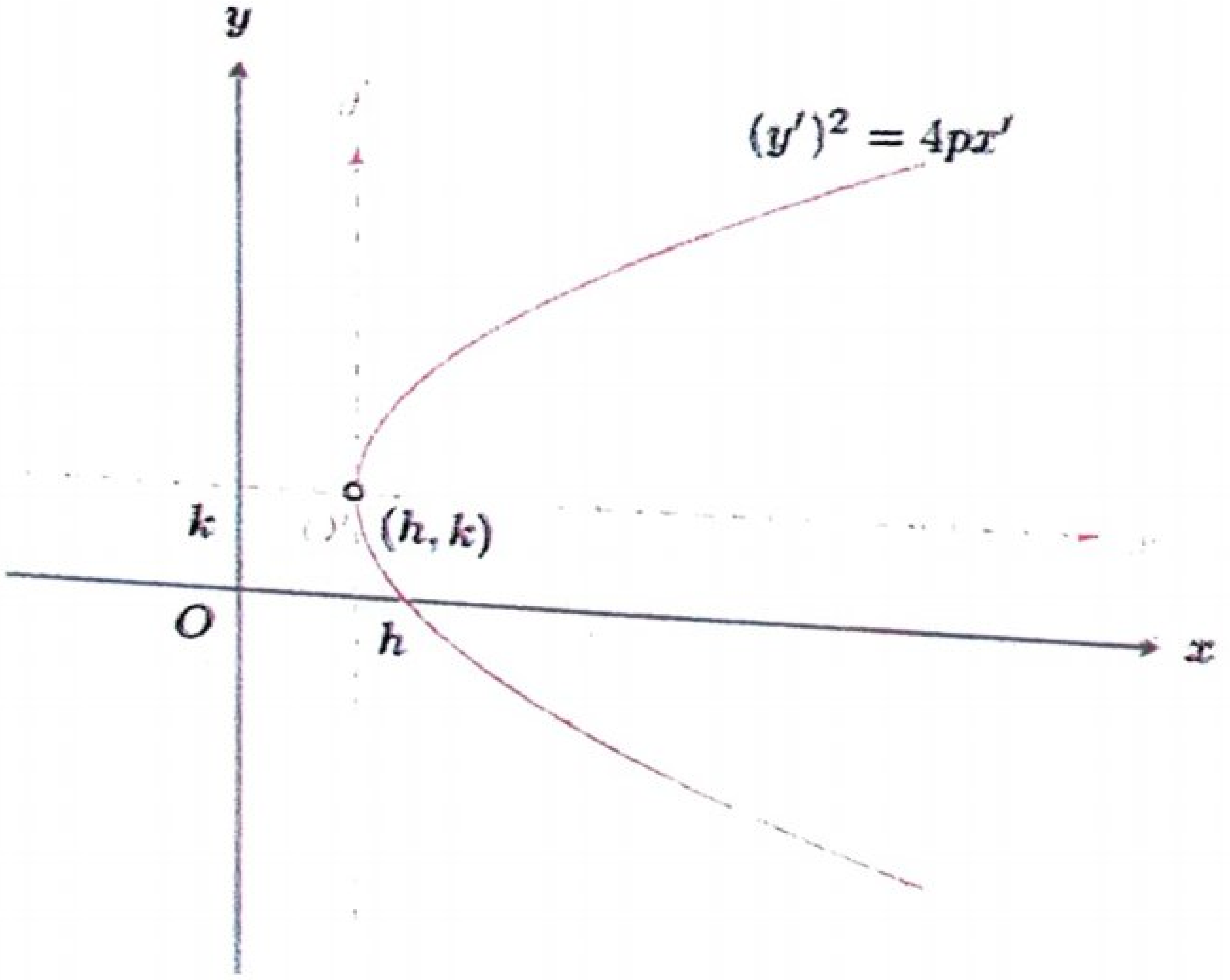
$$(x-h)^2 = \pm 4p(y-k).$$

Here the $(+)$ sign occurs if the parabola opens in the positive y direction and the $(-)$ sign occurs if the parabola opens in the negative y direction.

Parabola with Vertex (h, k) and Axis parallel to x-axis

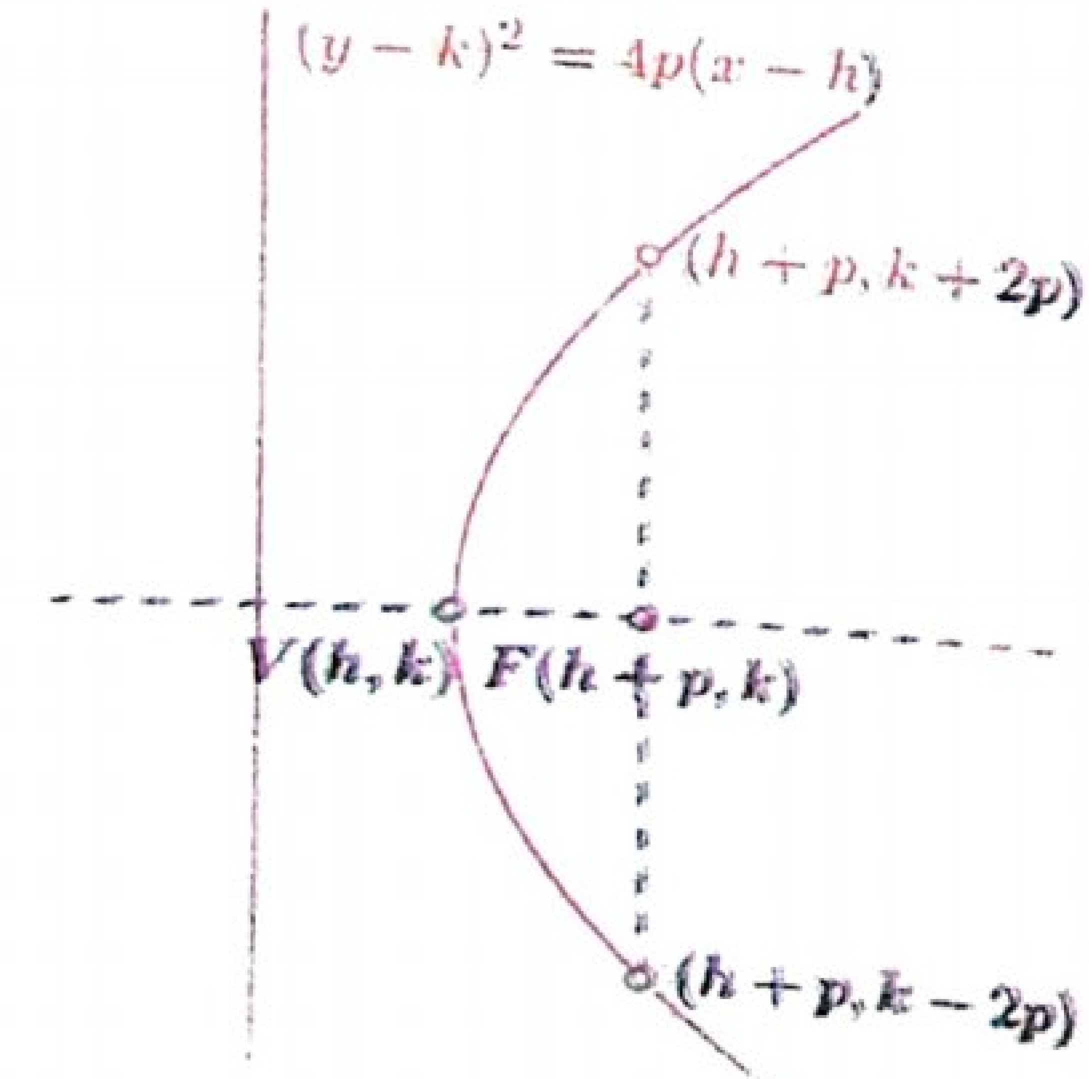
The equation of the parabola with vertex (h, k) and axis parallel to x-axis is of the form

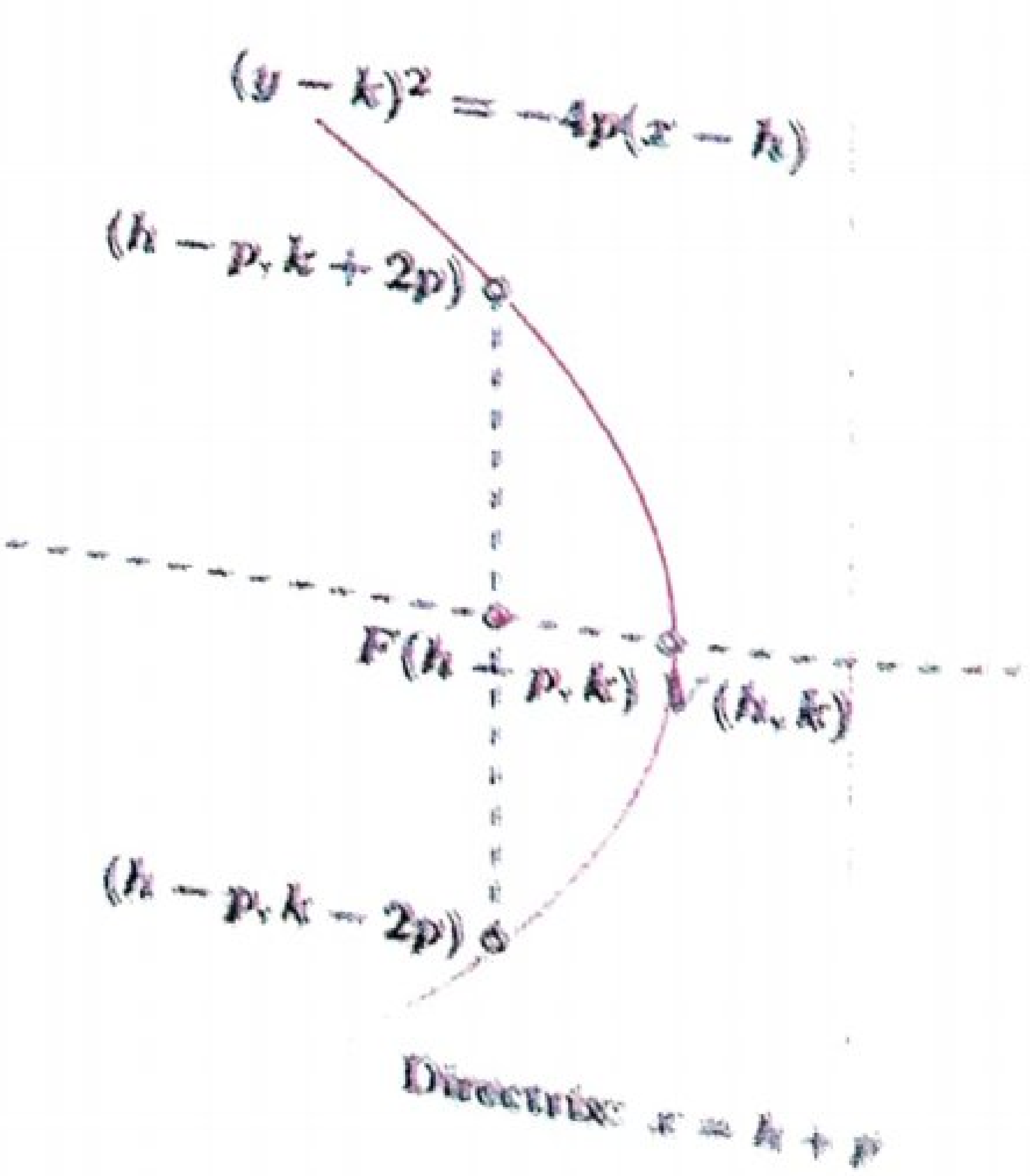
$$(y - k)^2 = \pm 4p(x - h).$$

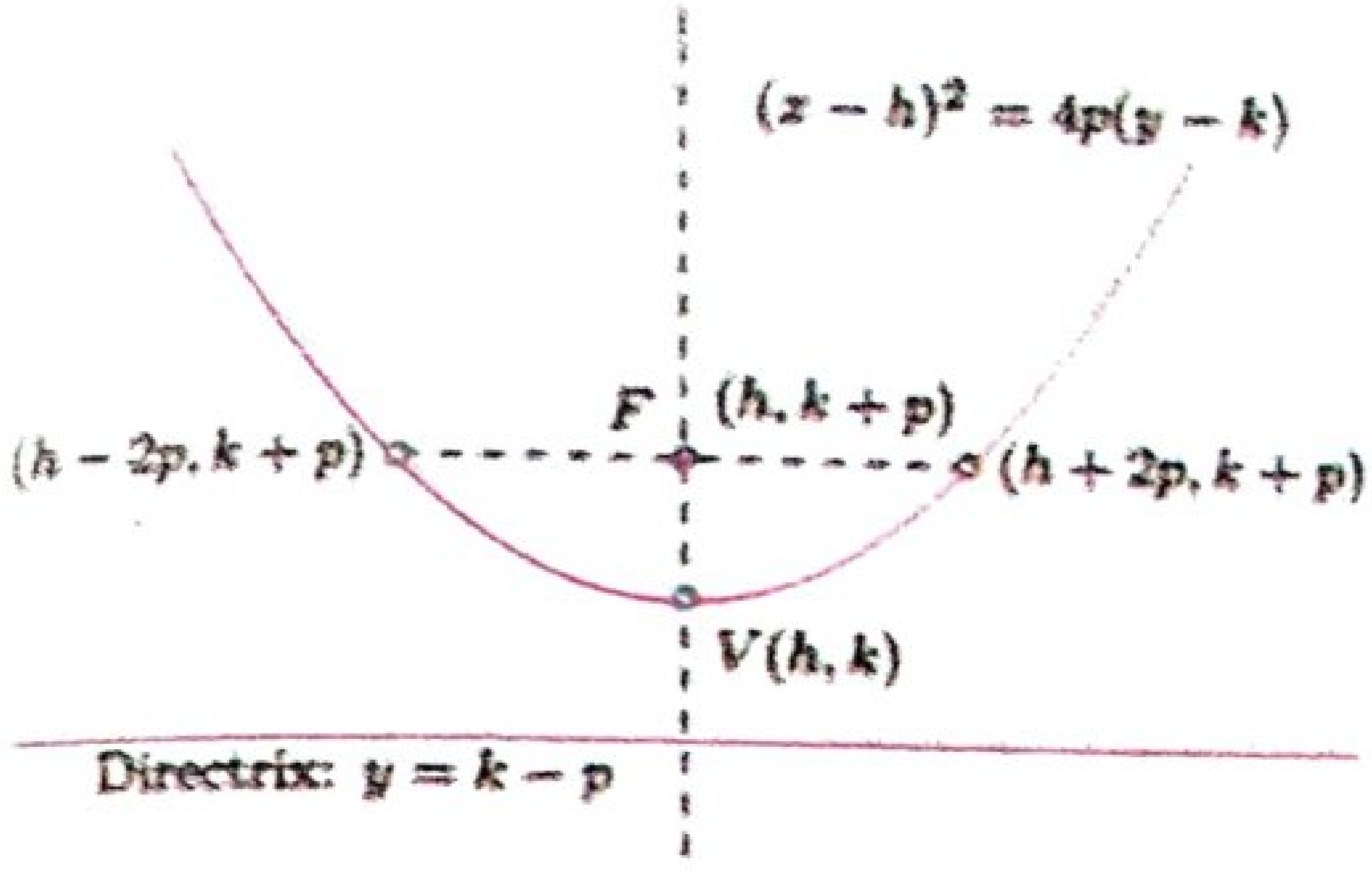

y
y'
$(y')^2 = 4px'$
k
O'
(h, k)
O
h
x

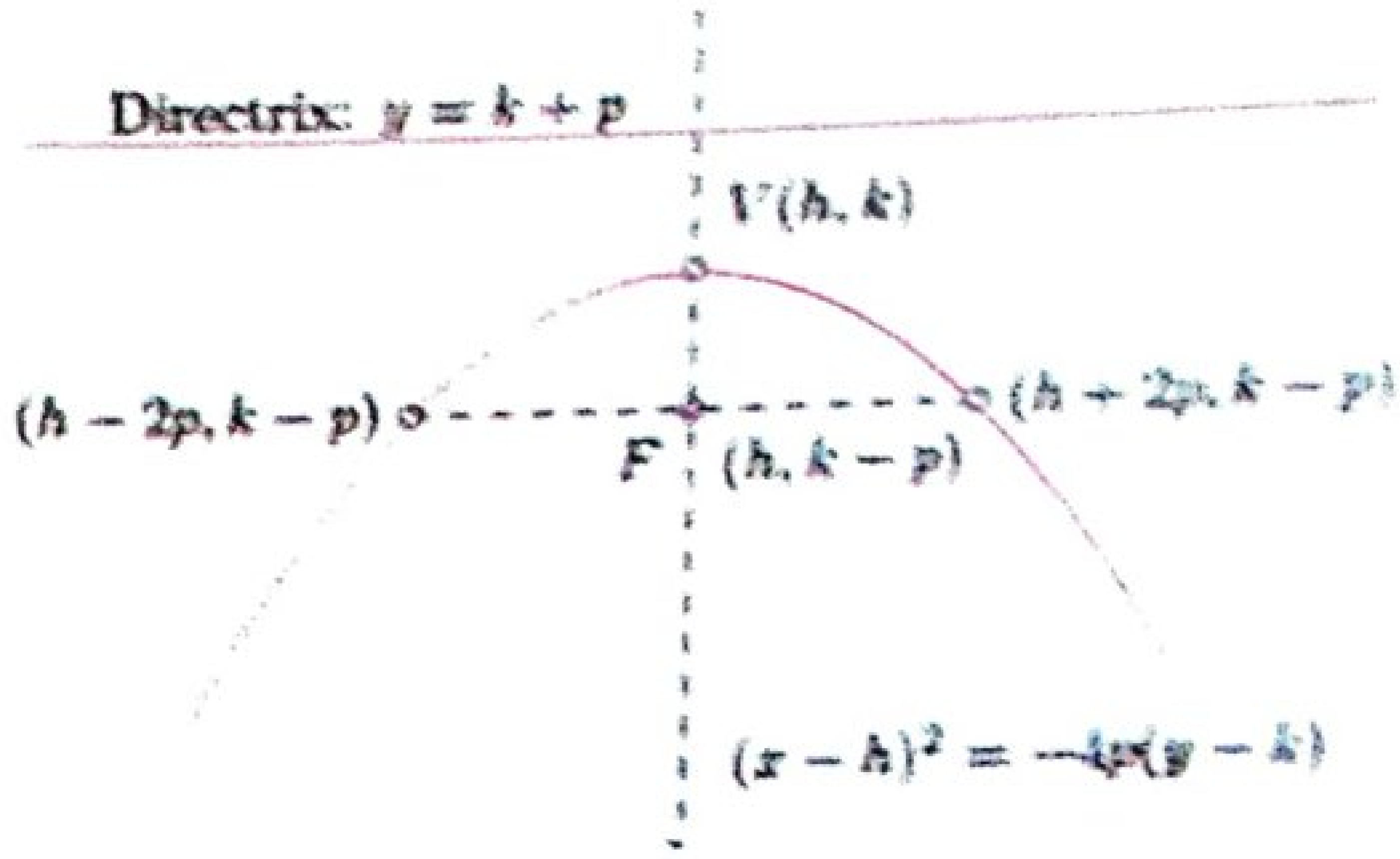
Here the $(+)$ sign occurs if the parabola opens in the positive x direction and the $(-)$ sign occurs if the parabola opens in the negative x direction.

The four possible such orientations whose vertices not at the origin are shown in the following figures. These are called the standard positions of a parabola, and the resulting equations are called the **standard forms of equation of a parabola**.


$(y - k)^2 = 4p(x - h)$
$(h + p, k + 2p)$
$V(h, k)$
$F(h + p, k)$
$(h + p, k - 2p)$
Directrix: $x = h - p$


$(y - k)^2 = -4p(x - h)$
$(h - p, k + 2p)$
$F(h - p, k)$
$V(h, k)$
$(h - p, k - 2p)$
Directrix: $x = h + p$


$(x-h)^2 = 4p(y-k)$
F
$(h, k+p)$
$(h-2p, k+p)$
$(h+2p, k+p)$
$V(h, k)$
Directrix: $y = k - p$


Directrix: $y = k + p$
$V(h, k)$
$(h-2p, k-p)$
$(h+2p, k-p)$
F
$(h, k-p)$
$(x-h)^2 = -4p(y-k)$

Equations of a Parabola:

Vertex at (h, k), Parallel to Coordinate Axis, $p > 0$

Equation	Vertex	Focus	Directrix	Axis of Symmetry	Description
$(y-k)^2 = 4p(x-h)$	(h, k)	$(h+p, k)$	$x = h - p$	horizontal axis, $y = k$	opens to the right
$(y-k)^2 = -4p(x-h)$	(h, k)	$(h-p, k)$	$x = h + p$	horizontal axis, $y = k$	opens to the left
$(x-h)^2 = 4p(y-k)$	(h, k)	$(h, k+p)$	$y = k - p$	vertical axis, $x = h$	opens up
$(x-h)^2 = -4p(y-k)$	(h, k)	$(h, k-p)$	$y = k + p$	vertical axis, $x = h$	opens down

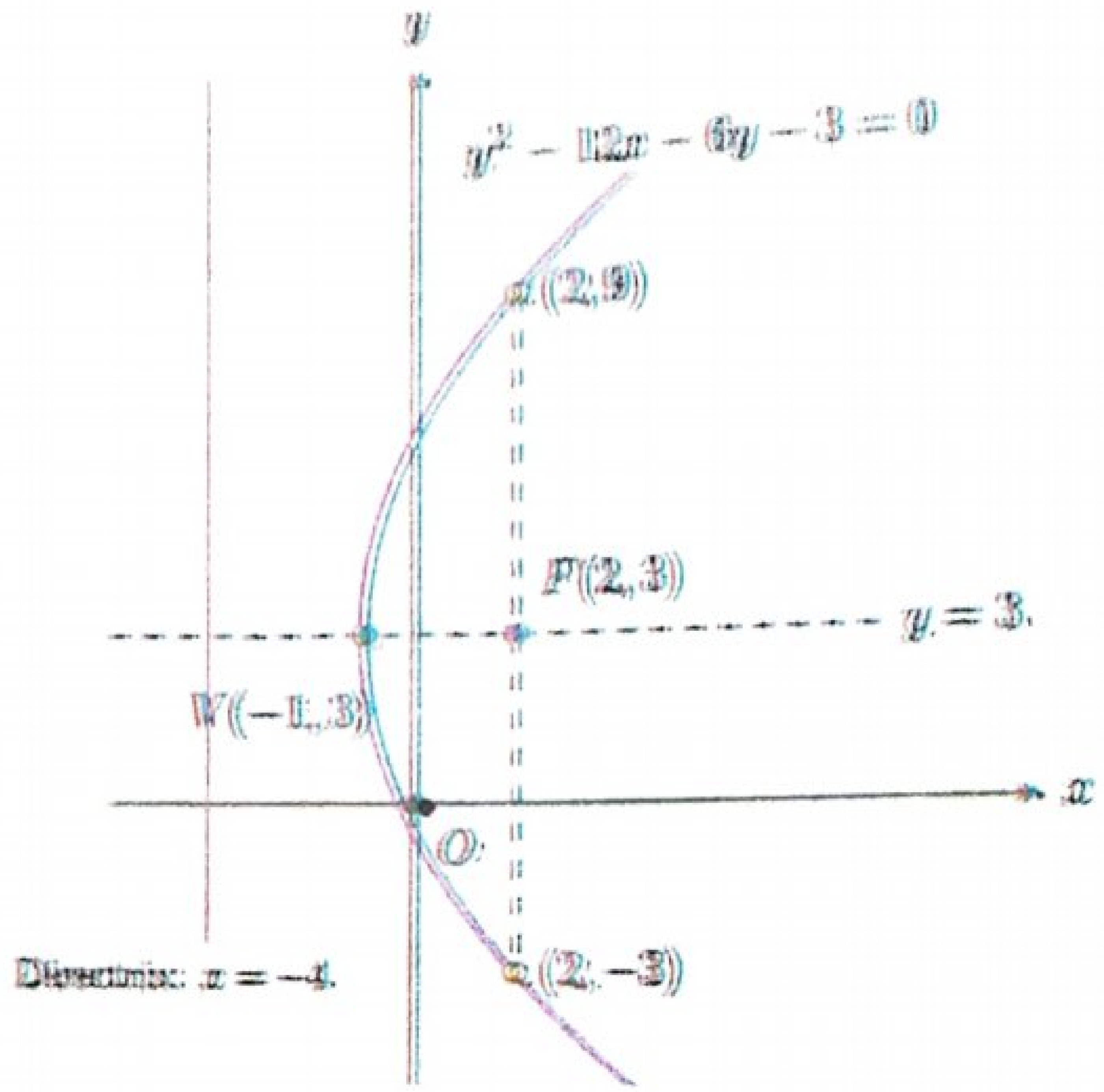
Example 8.

Write the given equation $y^2 - 12x - 6y - 3 = 0$ in standard form. Sketch its graph showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$\begin{aligned} y^2 - 12x - 6y - 3 &= 0 \\ y^2 - 6y &= 12x + 3 \\ y^2 - 6y + 3^2 &= 12x + 3 + 9 \\ (y-3)^2 &= 12x + 12 \\ (y-3)^2 &= 12(x+1) \end{aligned}$$

Comparing with $(y-k)^2 = 4p(x-h)$, we have $h = -1$, $k = 3$ and $p = 3$. Therefore, the vertex $V = (-1, 3)$, the focus $F = (2, 3)$ and the directrix is $x = -4$.



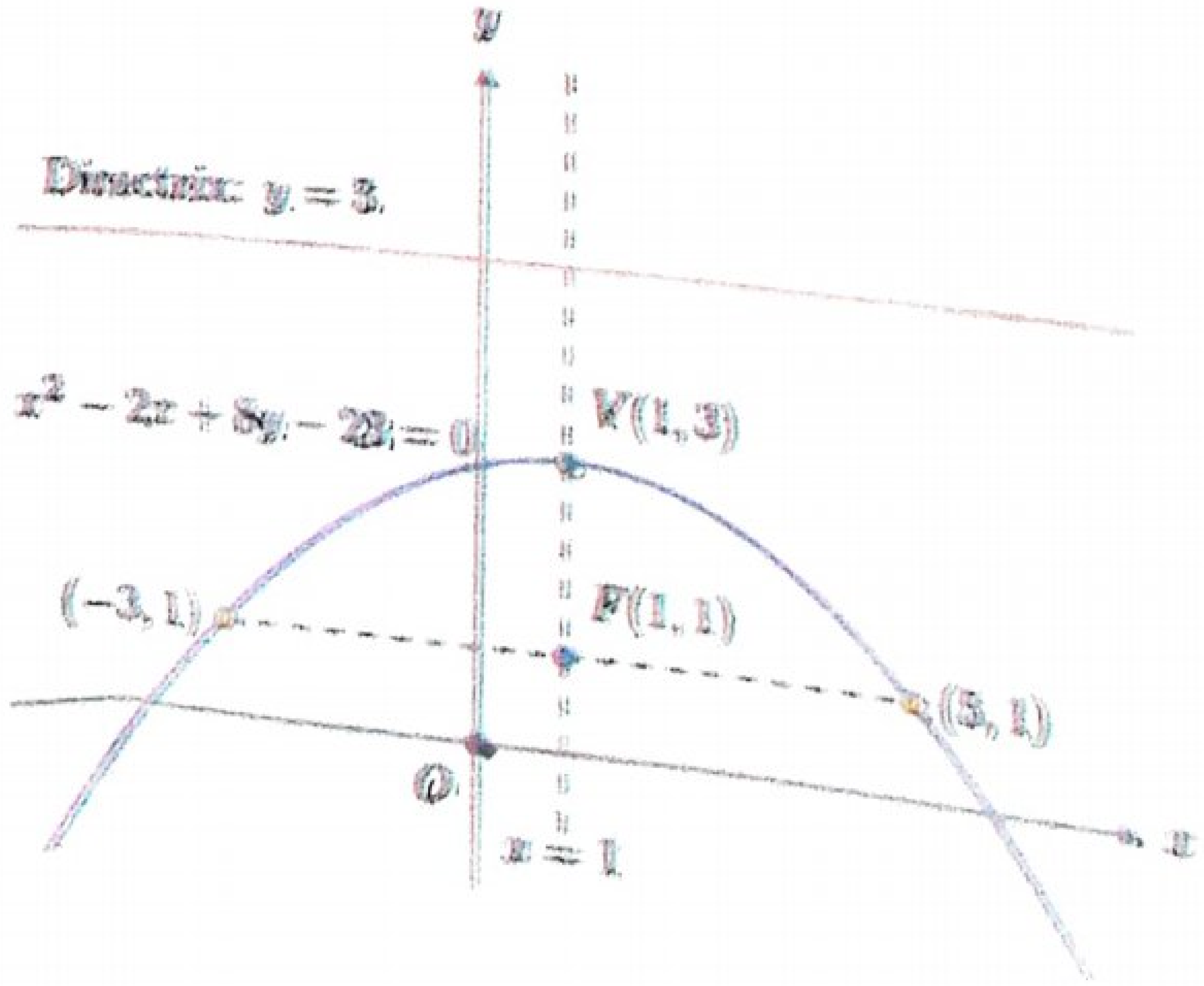
Example 9.

Write the given parabola equation $x^2 - 2x + 8y - 23 = 0$ in standard form. Sketch its graph showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$\begin{aligned} x^2 - 2x + 8y - 23 &= 0 \\ x^2 - 2x &= -8y + 23 \\ x^2 - 2x + 1^2 &= -8y + 23 + 1 \\ (x-1)^2 &= -8y + 24 \\ (x-1)^2 &= -8(y-3) \end{aligned}$$

Comparing with $(x-h)^2 = -4p(y-k)$, we have $h = 1$, $k = 3$ and $p = 2$.
Therefore, the vertex $V = (1, 3)$, the focus $F = (1, 1)$ and the directrix is $y = 5$.



Example 10.

Find the standard form of the equation of the parabola with vertex $(1,2)$ and focus at $(4,2)$.

Solution

Since the focus and vertex are on a horizontal line and the focus is to the right of the vertex, the parabola opens to the right.
The vertex $V=(1,2)$, the focus $F=(4,2)=(1+3,2)$.
Therefore $h=1$, $k=2$ and $p=3$.
The equation of the parabola is

$$\begin{aligned}(y-k)^2&=4p(x-h)\\(y-2)^2&=4\times 3(x-1)\\(y-2)^2&=12(x-1).\end{aligned}$$

Example 11.

Find the general form of the equation of the parabola with vertex $(2,1)$ and focus $(2,4)$.

Solution

Since the focus and vertex are on a vertical line and the focus is above the vertex, the parabola opens up.
Then $V=(h,k)=(2,1)$, $F=(2,4)=(2,1+3)$ and $p=3$.
The equation of the parabola is

$$\begin{aligned}(x-h)^2&=4p(y-k)\\(x-2)^2&=4\times 3(y-1)\\x^2-4x+16&=12y\\x^2-4x-12y+16&=0.\end{aligned}$$

The general form of the parabola is $x^2-4x-12y+16=0$.

Exercise 6.3

In Exercise 1 - 6, (a) sketch the parabola, (b) show the focus, vertex, directrix and end points of latus rectum:

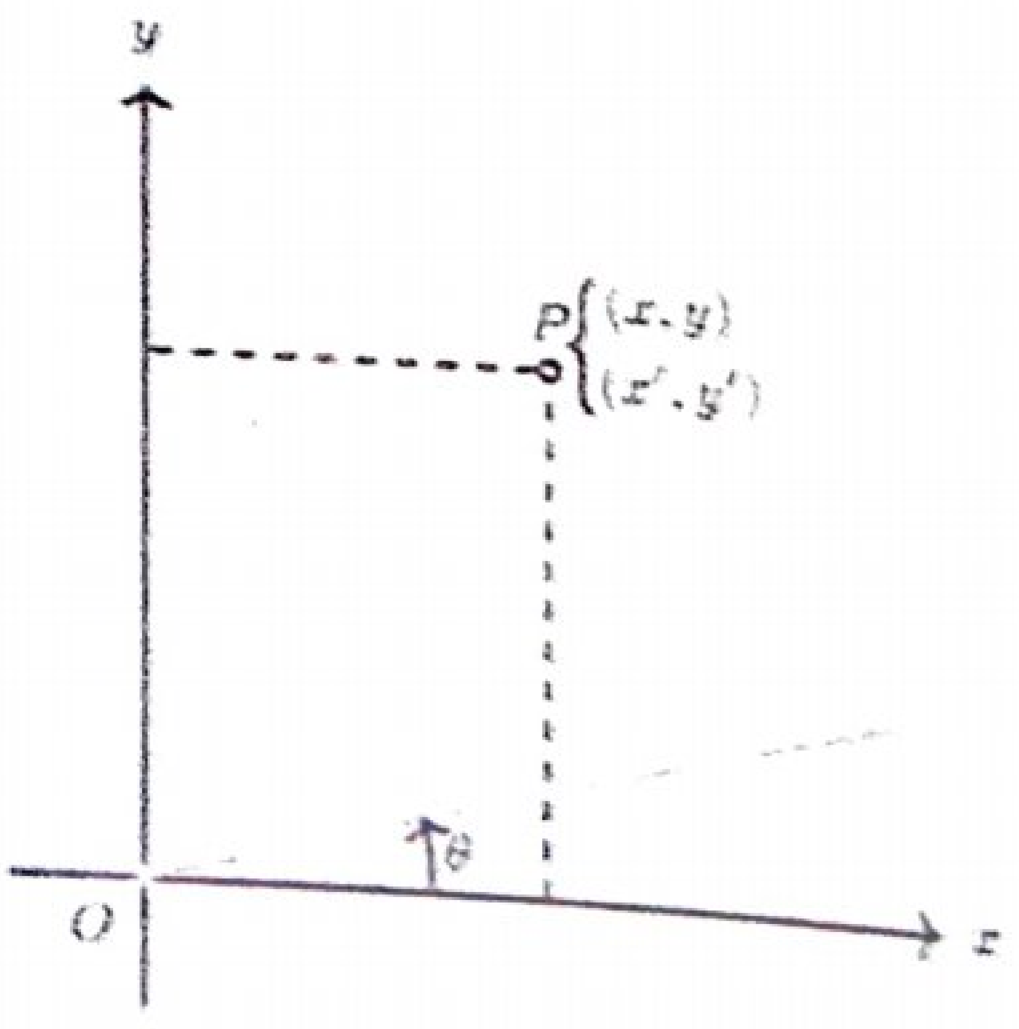
1. $x^2+4x+4y=0$
3. $(x+5)+(y-1)^2=0$
5. $x^2+4x+6y-2=0$
2. $(x-1)^2+8(y+1)=0$
4. $y^2+6y+8x+25=0$
6. $y^2+x+y=0$

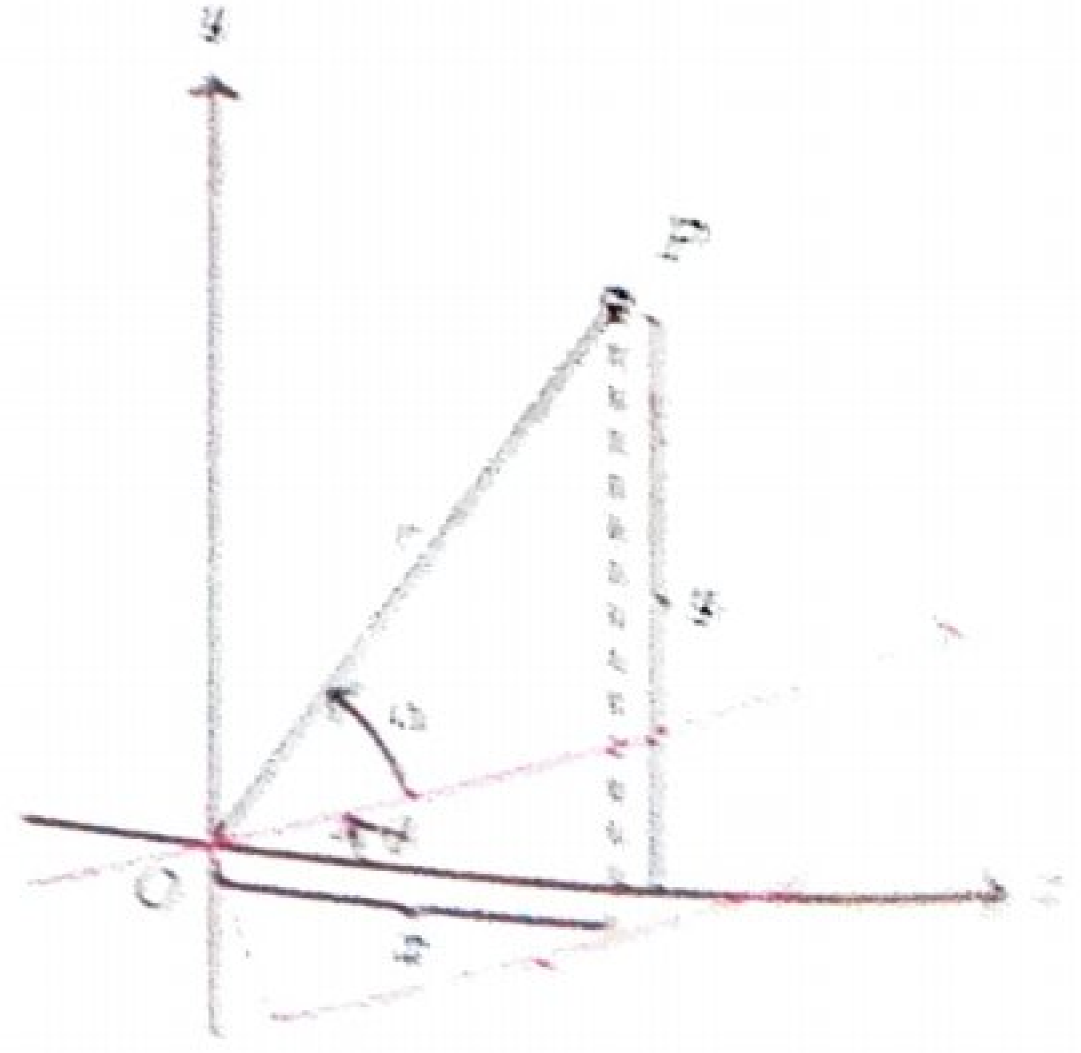
In Exercise 7 - 14, find an equation for the parabola satisfying the given conditions.

7. Vertex $(4, -5)$,
   focus $(1, -5)$.
8. Vertex $(5, 1)$,
   focus $(5, 3)$.
9. Vertex $(1, 1)$,
   directrix $y = -2$.
10. Vertex $(2, -1)$,
    directrix $x = -1$.
11. Focus $(-1, 4)$,
    directrix $x = 5$.
12. Focus $(6, 0)$,
    directrix $x = -6$.
13. Axis $x = 0$ passes through $(2, -1)$ and $(-4, 5)$.
14. Axis $y = 0$ passes through $(3, 2)$ and $(2, -3)$.

6.4 Rotation of Axes

In a rotation of axes, the origin remains fixed while the x-axis and the y-axis are rotated through an angle θ to a new position; the new positions of the x and y axes are denoted by x' and y' , respectively as shown in figure.





Here, the point P has the coordinates (x, y) relative to the xy-plane. We can express x and y in terms of x', y' and θ.

As in figure, r denotes the distance from the origin to the point P, and α denotes the angle between the positive x'-axis and the ray from O through P. Then, by using definitions of sine and cosine, we have

$$x' = r\cos\alpha, \quad y' = r\sin\alpha,$$

$$x = r\cos(\theta + \alpha), \quad y = r\sin(\theta + \alpha).$$

Next,

$$\begin{aligned} x &= r\cos(\theta+\alpha) \\ &= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ &= (r\cos\alpha)(\cos\theta) - (r\sin\alpha)(\sin\theta) \\ &= x'\cos\theta - y'\sin\theta \end{aligned}$$

Similarly,

$$\begin{aligned} y &= r\sin(\theta+\alpha) \\ &= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \\ &= (r\cos\alpha)(\sin\theta) + (r\sin\alpha)(\cos\theta) \\ &= x'\sin\theta + y'\cos\theta. \end{aligned}$$

Now we have the following theorem.

Theorem

If the x and y axes are rotated through an angle θ, then the coordinates (x, y) of a point P relatively to the xy-plane and the coordinates (x', y') of the same point relative to the new x' and y' axes are related by the formula

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta. \end{aligned}$$

Proof:
We have the rotation equations:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}$$

Multiplying with $\sin\theta$ and $\cos\theta$ to the respective rotation equations gives:

$$x\sin\theta = x'\sin\theta\cos\theta - y'\sin^2\theta \qquad (1)$$
$$y\cos\theta = x'\sin\theta\cos\theta + y'\cos^2\theta \qquad (2)$$

Solving equations (1) and (2)

$$y' = -x\sin\theta + y\cos\theta$$

Multiplying with $\cos\theta$ and $\sin\theta$ to the respective rotation equations gives:

$$x\cos\theta = x'\cos^2\theta - y'\sin\theta\cos\theta \qquad (3)$$
$$y\sin\theta = x'\sin^2\theta + y'\sin\theta\cos\theta \qquad (4)$$

Adding equations (3) and (4) gives

$$x' = x\cos\theta + y\sin\theta$$

So, we have the formulas

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta.$$

Example 12.

Find the new coordinates of the point (1, 2) if the coordinate axes are rotated through an angle of $\theta = 30°$

Solution

We have $x = 1$, $y = 2$, $\cos\theta = \cos 30° = \frac{\sqrt{3}}{2}$, $\sin\theta = \sin 30° = \frac{1}{2}$.
Then

$$x' = x\cos\theta + y\sin\theta$$
$$x' = 1(\frac{\sqrt{3}}{2}) + 2(\frac{1}{2}) = \frac{\sqrt{3}}{2} + 1$$
$$y' = -x\sin\theta + y\cos\theta$$
$$y' = -1(\frac{1}{2}) + 2(\frac{\sqrt{3}}{2}) = -\frac{1}{2} + \sqrt{3}.$$

Therefore, the new coordinates are $(\frac{\sqrt{3}}{2} + 1, -\frac{1}{2} + \sqrt{3})$.

Eliminating the $x'y'$-Term

We have the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with $B \neq 0$.

We substitute

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

in the general equation, then

$$A'(x')^2 + B'x'y' + C'(y')$$

here

$$
\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta,\\
B' &= B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta,\\
&= B(\cos 2\theta) - (A - C)\sin 2\theta,\\
C' &= A\sin\theta - B\sin\theta\cos\theta + C\cos^2\theta,\\
D' &= D\cos\theta + E\sin\theta,\\
E' &= -D\sin\theta + E\cos\theta,\\
F' &= F.
\end{aligned}
$$

eliminate the $x'y'$-term, choose θ such that $B' = 0$.

$$B(\cos 2\theta) - (A - C)\sin 2\theta = 0$$

$$\cot 2\theta = \frac{A - C}{B}.$$

Note

1. If $\cot 2\theta = 0$, then $2\theta = 90°$ so that $\theta = 45°$.
2. If $\cot 2\theta \neq 0$, first find $\cos 2\theta$ to obtain the correct acute angle θ.

hen we have

$$\boxed{\text{if } B' = 0\text{ , then } \cot 2\theta = \frac{A - C}{B}\text{ .}}$$

can prove that

$$B'^2 - 4A'C' = B^2 - 4AC,$$

d since $B' = 0$, we have

$$B^2 - 4AC = -4A'C'.$$

$B^2 - 4AC = 0$, then $-4A'C' = 0$ and we have
$A' = 0$ or $C' = 0$.
hen $A' = 0$, $C'y'^2 + D'x' + E'y' + F' = 0$.
hen $C' = 0$, $A'x'^2 + D'x' + E'y' + F' = 0$.

s means that if $B^2 - 4AC = 0$, after rotation with θ, then the general ation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ becomes the standard form of the ation of a parabola.

rotation of the coordinate axes through an angle θ transforms the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into the form

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$
2. $A + C = A' + C'$
3. $B^2 - 4AC = B'^2 - 4A'C'$

Example 13.

Rotate the coordinate axes to remove the xy-term of the equation

$$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$$

in $x'y'$- coordinate system. Then sketch the graph.

Solution

$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

Comparing with the general equation, we get

$A = 3,\ B = -2\sqrt{3},\ C = 1.$

Therefore $B^2 - 4AC = (-2\sqrt{3})^2 - 4(3)(1) = 0$ and

$$\cot 2\theta = \frac{A - C}{B} = \frac{3 - 1}{-2\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

Thus, $2\theta = 120^\circ$ and so $\theta = 60^\circ$.

Next,

$$\sin\theta = \sin 60^\circ = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\theta = \cos 60^\circ = \frac{1}{2}.$$

By using $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, we get

$$x = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' \quad \text{and} \quad y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'.$$

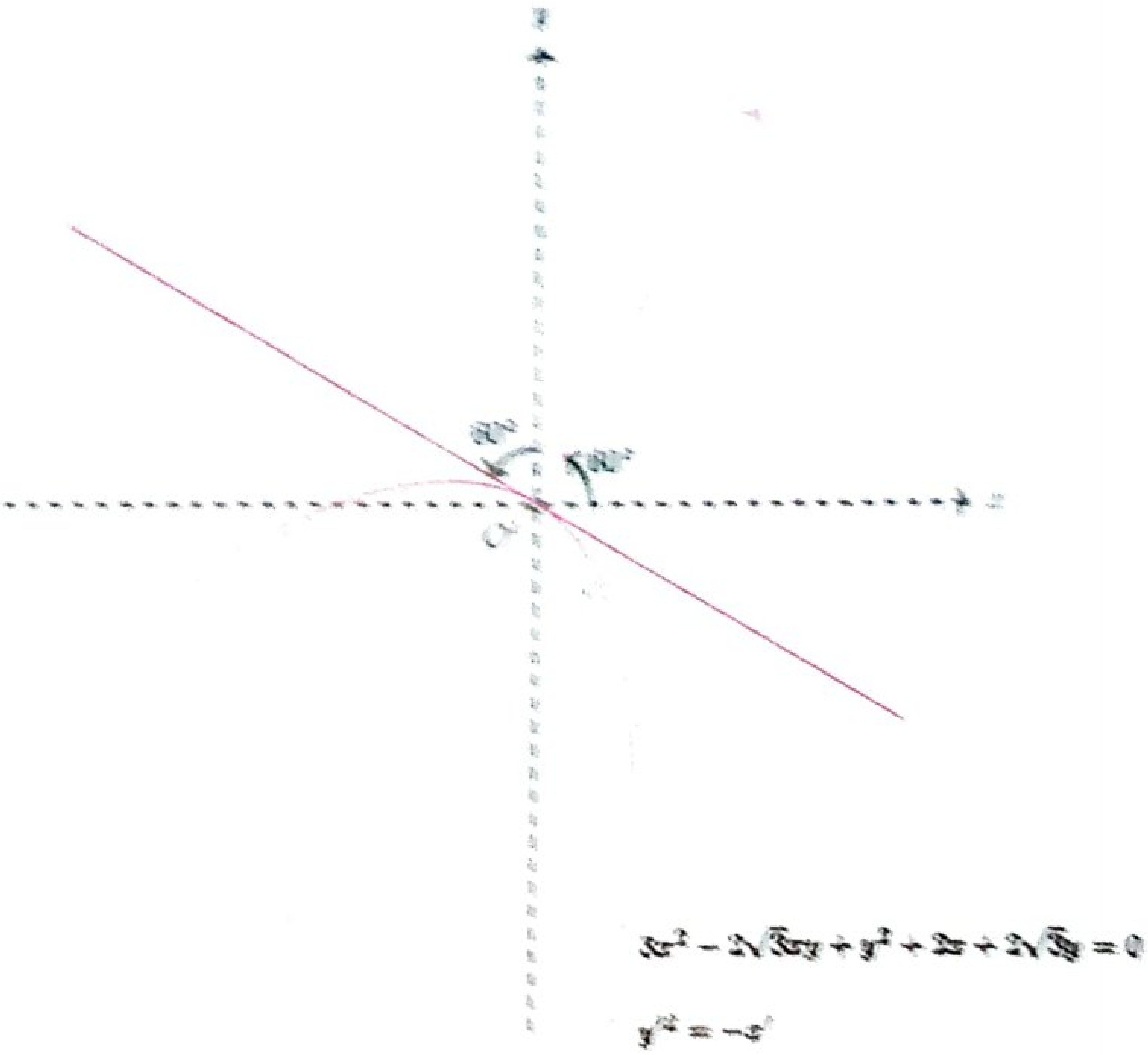
By substituting $x = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y'$ and $y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$ in given equation we

$$\frac{3}{4}((x')^2 - 2\sqrt{3}x'y' + 3(y')^2) - \frac{\sqrt{3}}{2}(x' - \sqrt{3}y')(\sqrt{3}x' + y')$$
$$-\frac{1}{4}(3(x')^2 + 2\sqrt{3}x'y' + (y')^2) + x' - \sqrt{3}y' + 3x' + \sqrt{3}y' = 0$$

then

$$(y')^2 = -x'.$$

see that the new conic is a parabola with vertex $(0,0)$.



Example 14.

Rotate the coordinate axes to remove the xy-term of the equation

$$16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$$

$x'y'$-coordinate system. Then sketch the graph.

Solution

$16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$

Comparing with the general equation, we get

$A = 16,\ B = -24,\ C = 9.$

We find that $B^2 - 4AC = (-24)^2 - 4(16)(9) = 0$ and

$$\cot 2\theta = \frac{A - C}{B} = \frac{16 - 9}{-24} = -\frac{7}{24}$$

Using trigonometric identities, we see that $\cos 2\theta = -\frac{7}{25}$.

Therefore,

$$\sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}.$$

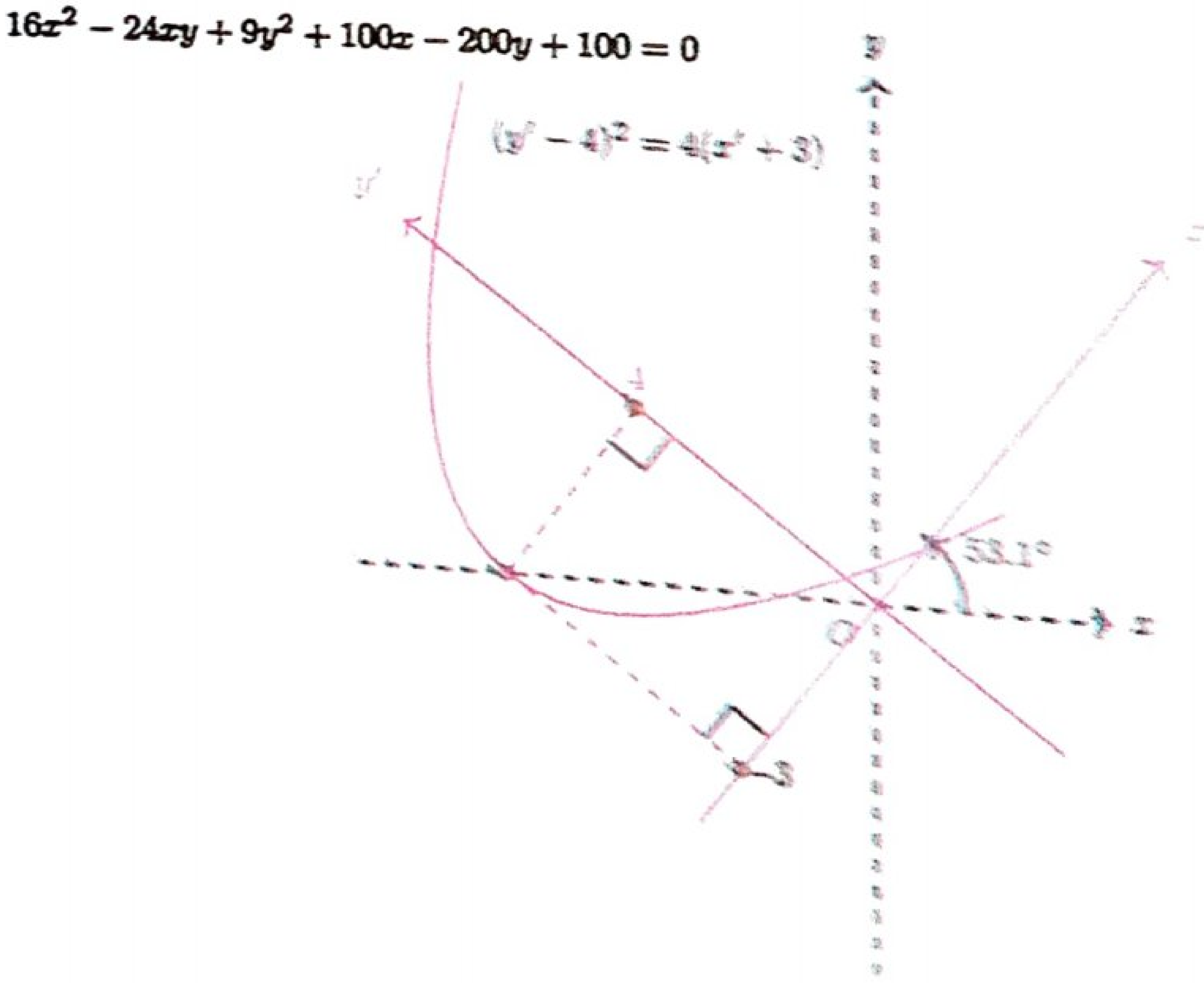
We get $x = \frac{3}{5}x' - \frac{4}{5}y'$ and $y = \frac{4}{5}x' + \frac{3}{5}y'$.

Substituting these values into the given equation, we get

$$\begin{aligned} 25(y')^2 - 100x' - 200y' + 100 &= 0 \\ (y')^2 - 8y' &= 4x' - 4 \\ (y'-4)^2 &= 4x' + 12 \\ (y'-4)^2 &= 4(x'+3). \end{aligned}$$

We see that the new conic in $x'y'$-plane is a parabola with vertex $(-3, 4)$.

Since $\sin\theta = \frac{4}{5}$, we find that $\theta = 53.1°$.



Exercise 6.4

Rotate the coordinate axes to remove the xy-term. Then sketch the graph.

1. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$.
2. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$.
3. $x^2 - 2xy + y^2 - 20x - 20y + 100 = 0$.
4. $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$.
5. $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$.

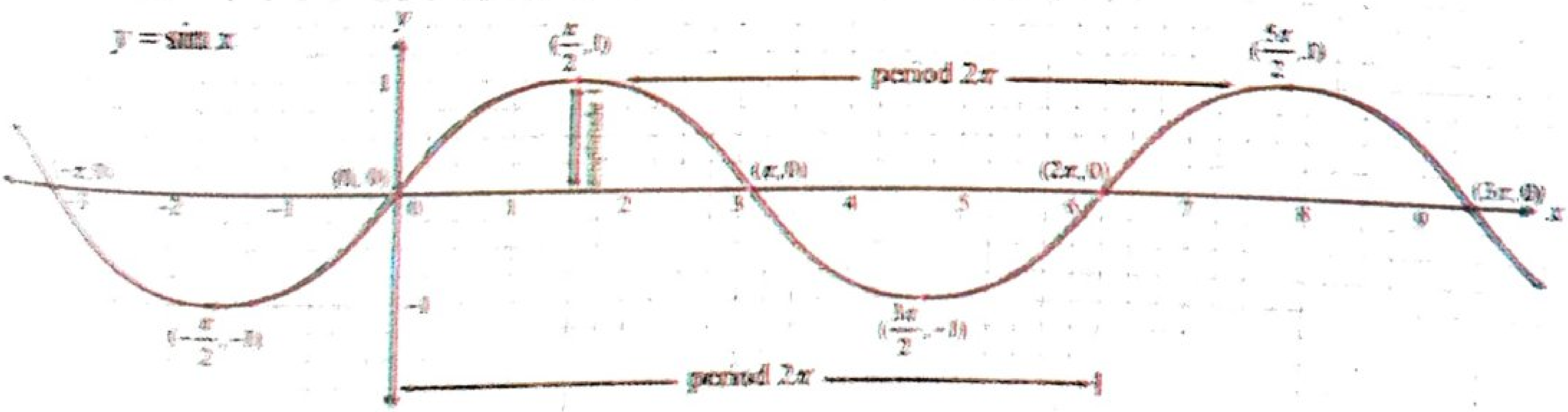
Chapter 7

Trigonometric Functions

This chapter starts with graphs of sine functions. Then graphs of cosine, tangent and other trigonometric functions are shown. After that, the inverse of those functions and their graphs come. The essential part of this chapter is the differentiation of trigonometric functions.

7.1 Graphs of Sine Functions

Graph of the Sine Function $y = \sin x$



Domain: The set $\mathbb{R}$ of all real numbers Range: $\{y \mid -1 \le y \le 1\}$

$$\underline{\text{Five key points}}: \begin{cases} x\text{-intercepts: } (0,0), (\pi,0), (2\pi,0) \\ \text{maximum point: } (\frac{\pi}{2}, 1) \qquad \text{minimum point: } (\frac{3\pi}{2}, -1) \end{cases}$$

Periodic function: If $\boxed{f(x) = f(x+p)}$ where p is a positive real number, then the function f is called a **periodic function**. If p is the smallest such number, then p is the **period** of function f. The sine function $y = \sin x$ is a periodic function with period 2π, since 2π is the smallest positive real number such that $\sin x = \sin(x + 2\pi)$.

The **amplitude** of a periodic function is half the difference between the maximum and minimum values. So the amplitude of the sine function $y = \sin x$ is $\dfrac{1-(-1)}{2} = 1$.

Graph of the Sine Function $y = a\sin bx,\ a > 0, b > 0$

In the following example we will see how to obtain the graph of $y = a\sin bx$, where $a > 0$ and $b > 0$, from the graph of $y = \sin x$.

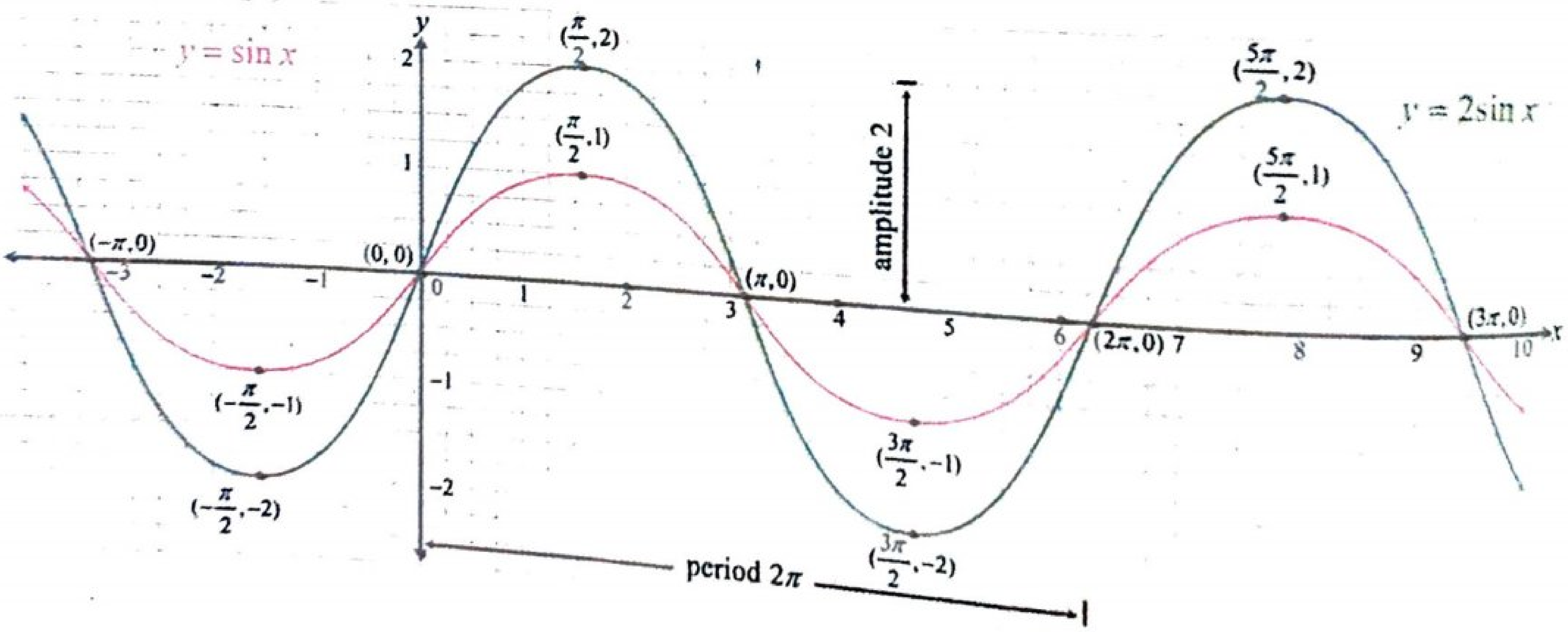
Example 1.

From the graph of $y = \sin x$, draw step-by-step transformation graphs to get the graph of $y = 2\sin\frac{\pi}{2}x$.

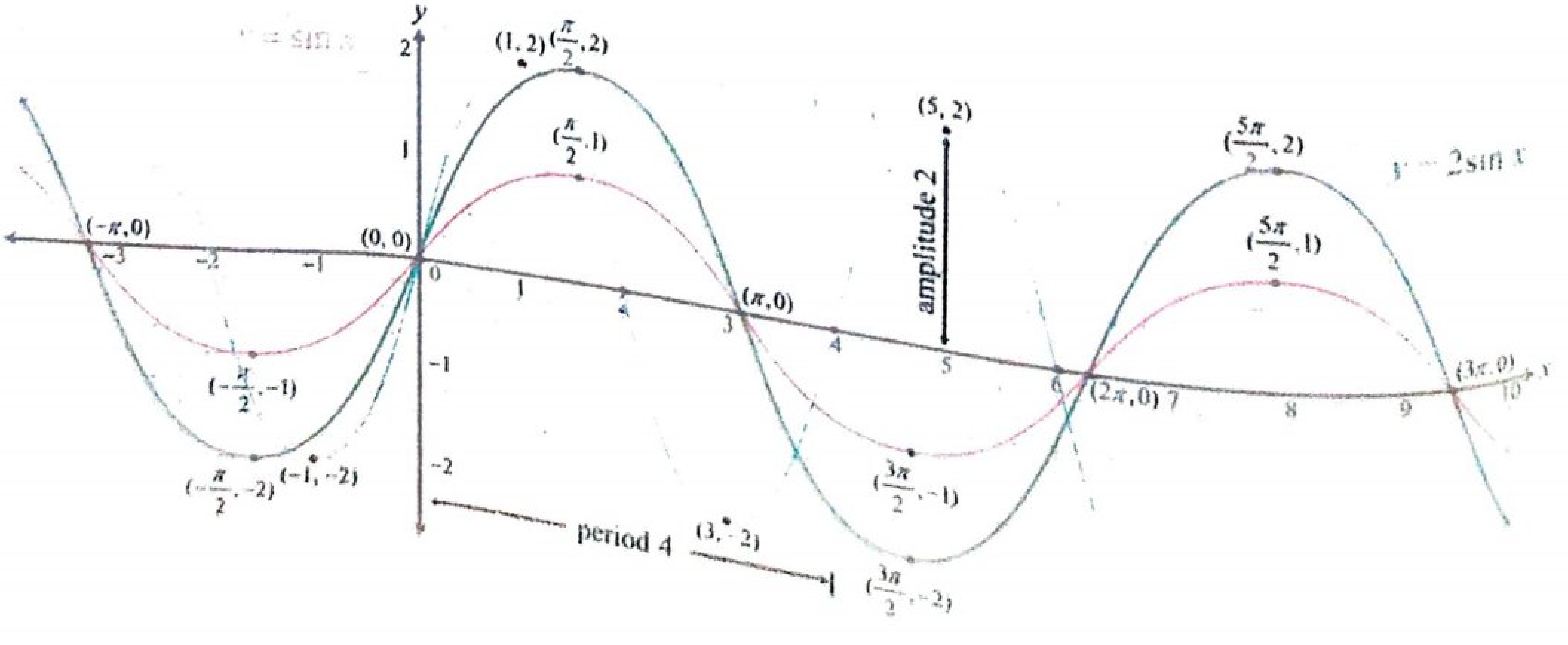
Solution

Method 1

$$y = \sin x \xrightarrow[\text{scale factor 2}]{\text{vertical scaling}} y = 2\sin x$$

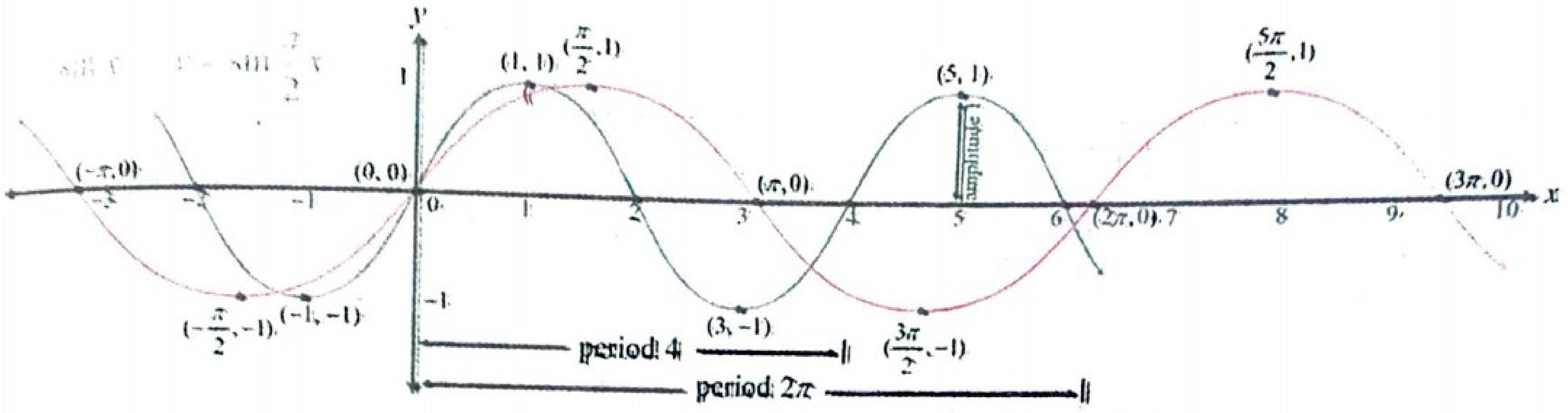


$$y = \sin x \xrightarrow[\text{scale factor 2}]{\text{vertical scaling}} y = 2\sin x \xrightarrow[\text{scale factor}\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi}]{\text{horizontal scaling}} y = 2\sin\frac{\pi}{2}x$$

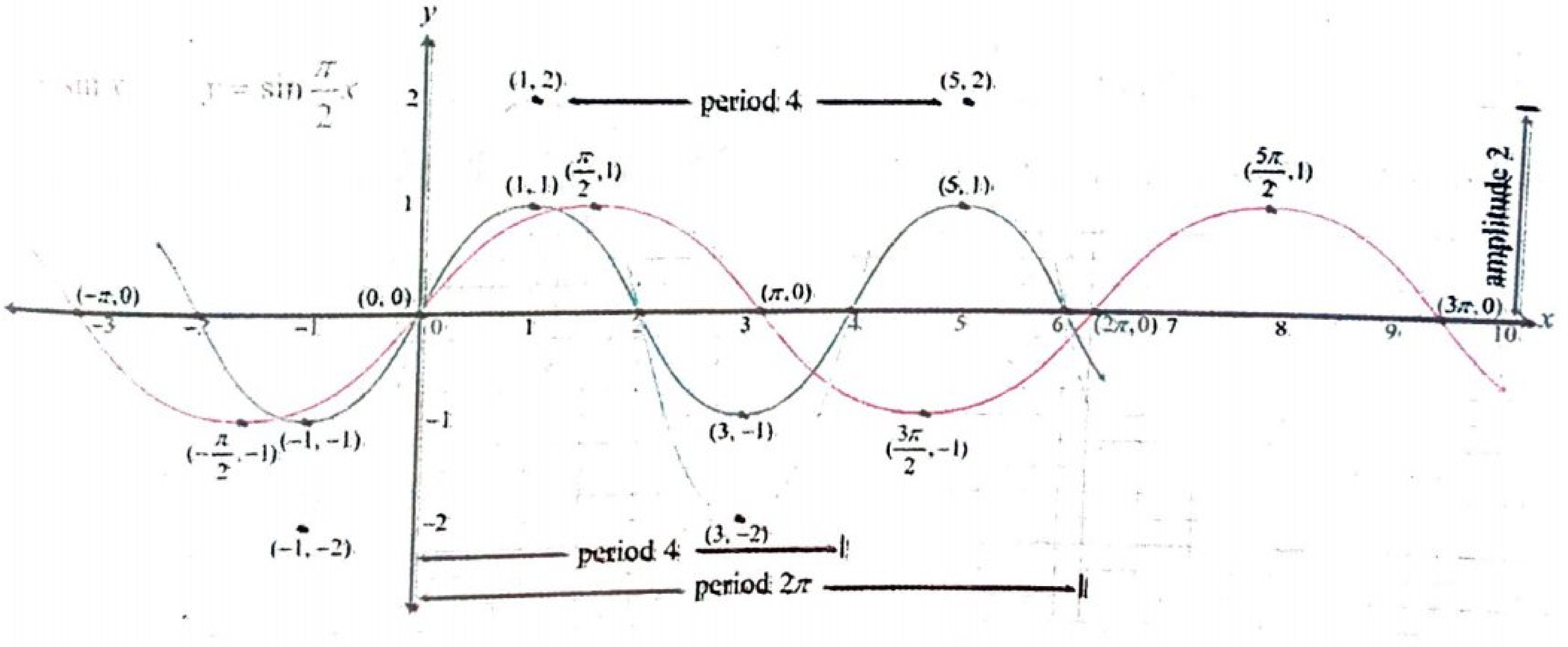


Method 2

$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} y = \sin \frac{\pi}{2}x$$



$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} y = \sin \frac{\pi}{2}x \xrightarrow[\text{scale factor } 2]{\text{vertical scaling}} y = 2\sin \frac{\pi}{2}x$$



From the graph of $y = \sin x$, the graph of $y = a\sin bx$, $a > 0, b > 0$, can be obtained as

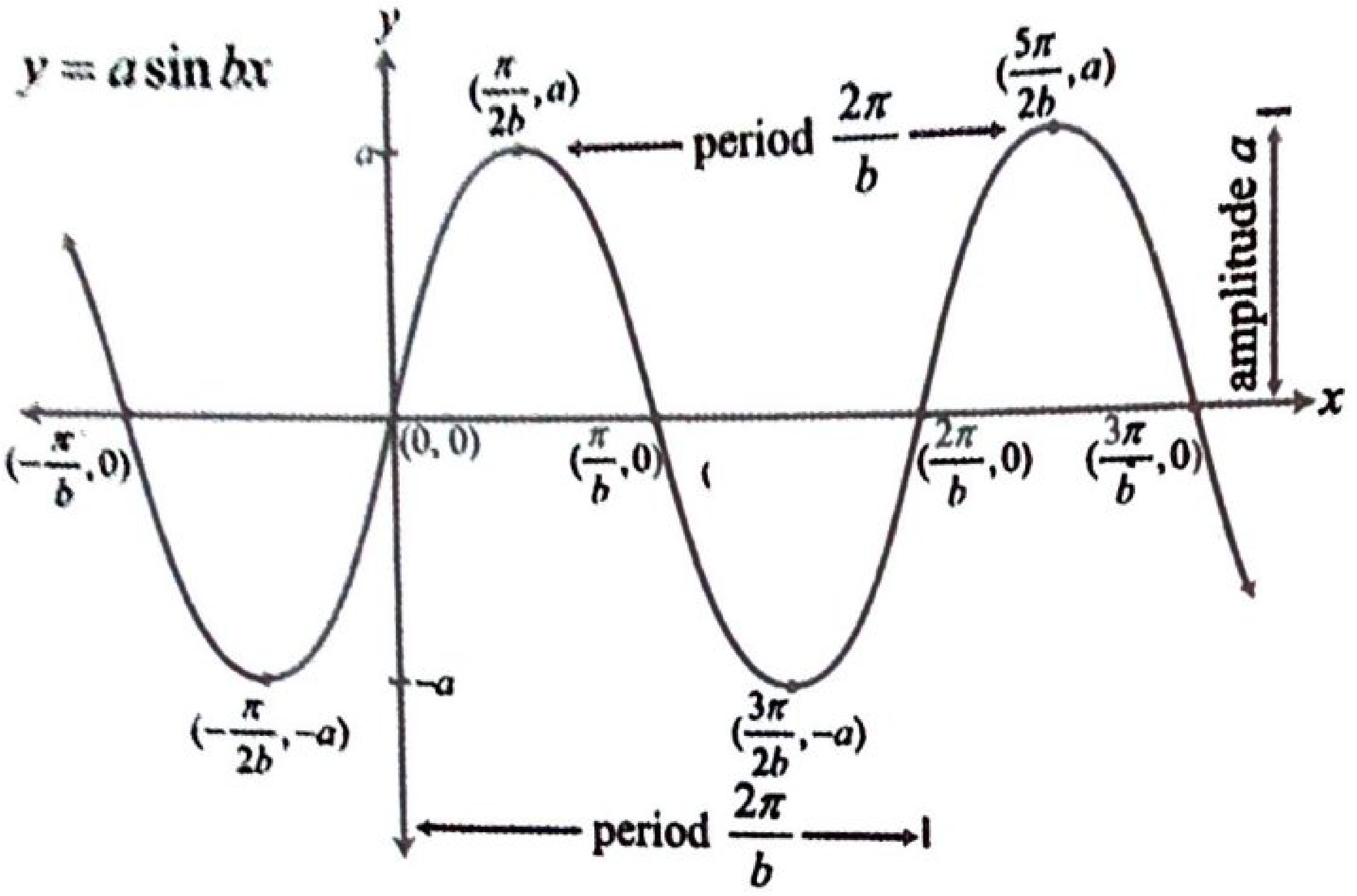
$$y = \sin x \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} y = a\sin x \xrightarrow[\text{scale factor } \frac{1}{b}]{\text{horizontal scaling}} y = a\sin bx$$

or

$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{b}]{\text{horizontal scaling}} y = \sin bx \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} y = a\sin bx$$

$$y = \sin x \to y = a\sin bx$$

$$(x, y) \to (\frac{x}{b}, ay)$$



Domain: $\mathbb{R}$

Range: $\{y \mid -a \leq y \leq a\}$

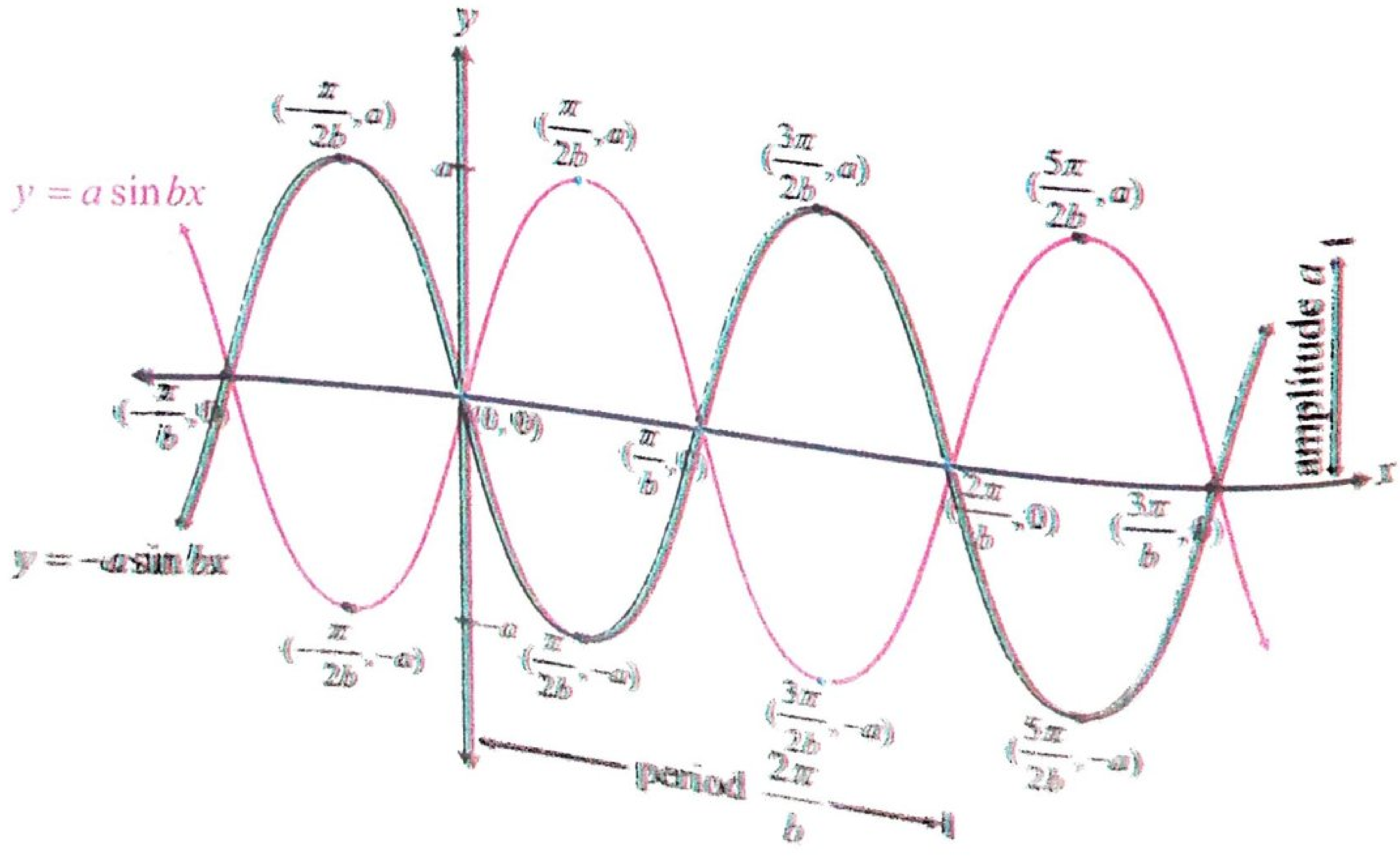
Period: $\dfrac{2\pi}{b}$

Amplitude: a

$$\underline{\text{Five key points}}: \begin{cases} x\text{-intercepts: } (0,0), (\frac{\pi}{b}, 0), (\frac{2\pi}{b}, 0) \\ \text{maximum point: } (\frac{\pi}{2b}, a) \qquad \text{minimum point: } (\frac{3\pi}{2b}, -a) \end{cases}$$

Graph of the Sine Function $y = -a\sin bx$, $a > 0, b > 0$

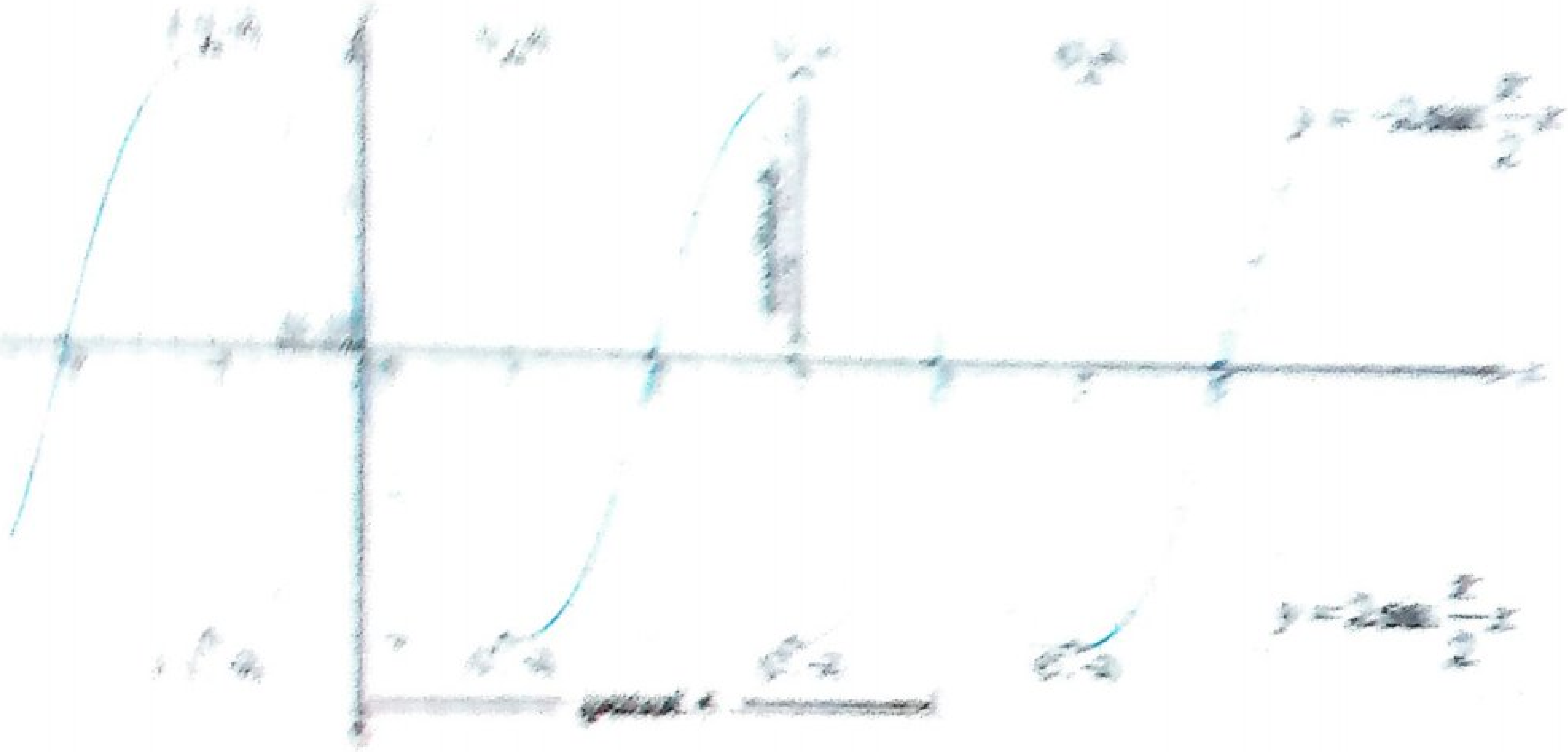
$$y = a\sin bx \xrightarrow{\text{Reflection on the } x\text{-axis}} y = -a\sin bx$$



Example 4

Draw the graphs of $y = 2\sin\frac{\pi}{2}x$ and $y = -2\sin\frac{\pi}{2}x$.

Solution



Note that, $y = a\sin(-b)x = -a\sin bx$, so no need to consider $b < 0$.

Graph of the Sine Function $y = a\sin b(x - h) + k$

$$y = a\sin bx \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} y = a\sin b(x - h) + k$$

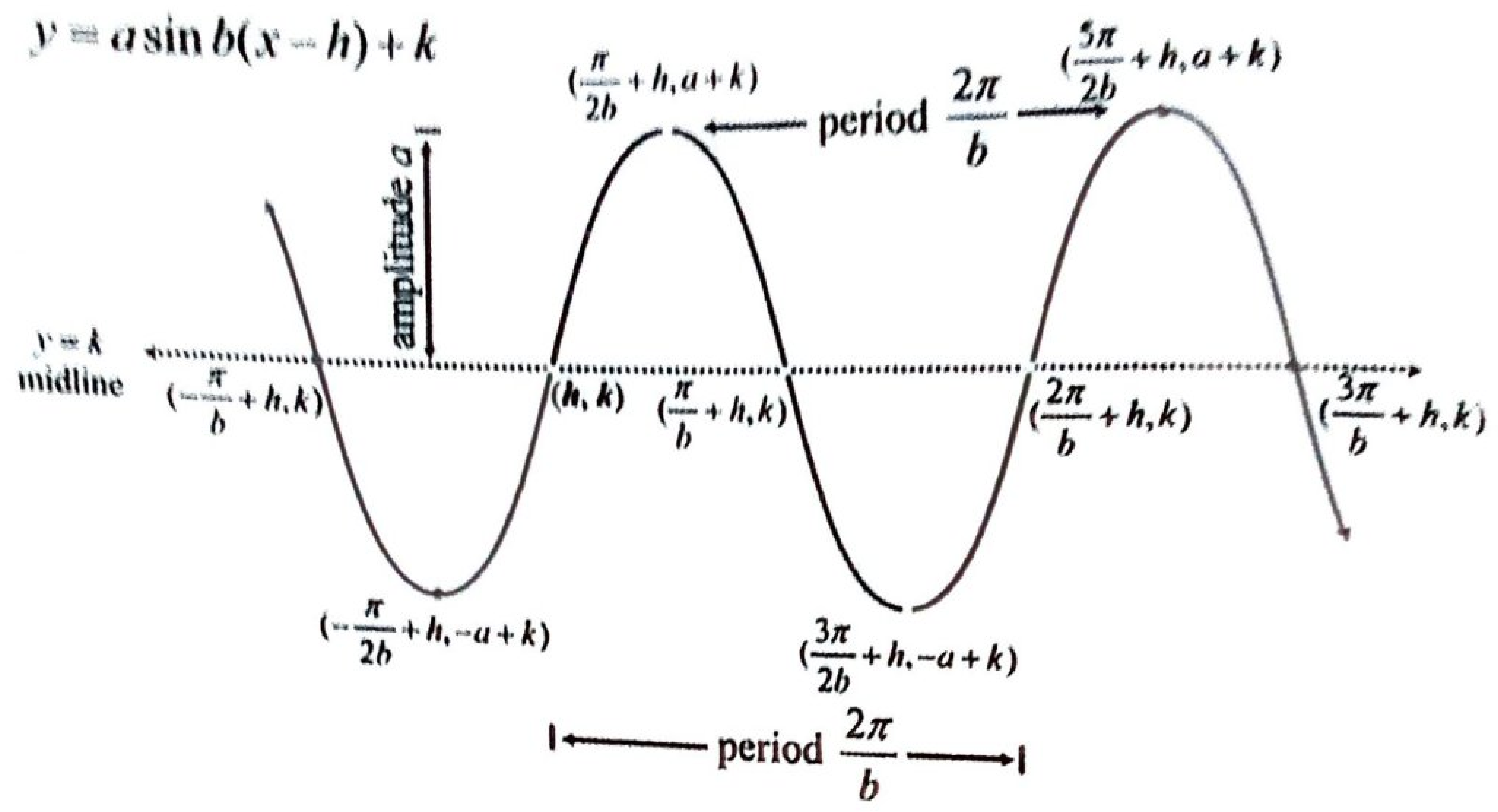
$$y = \sin x \to y = a\sin bx \to y = a\sin b(x - h) + k$$
$$(x, y) \to \left(\frac{x}{b}, ay\right) \to \left(\frac{x}{b} + h, ay + k\right)$$

Key points on the midline $y = k$

- $(0, 0) \to (0, 0) \to (h, k)$
- $(\pi, 0) \to \left(\frac{\pi}{b}, 0\right) \to \left(\frac{\pi}{b} + h, k\right)$
- $(2\pi, 0) \to \left(\frac{2\pi}{b}, 0\right) \to \left(\frac{2\pi}{b} + h, k\right)$

Maximum and minimum points

- $\left(\frac{\pi}{2}, 1\right) \to \left(\frac{\pi}{2b}, a\right) \to \left(\frac{\pi}{2b} + h, a + k\right)$
- $\left(\frac{3\pi}{2}, -1\right) \to \left(\frac{3\pi}{2b}, -a\right) \to \left(\frac{3\pi}{2b} + h, -a + k\right)$


$y = a\sin b(x-h)+k$
$(\frac{\pi}{2b}+h, a+k)$
period $\frac{2\pi}{b}$
$(\frac{5\pi}{2b}+h, a+k)$
amplitude a
$y = k$ midline
$(-\frac{\pi}{b}+h, k)$
(h, k)
$(\frac{\pi}{b}+h, k)$
$(\frac{2\pi}{b}+h, k)$
$(\frac{3\pi}{b}+h, k)$
$(-\frac{\pi}{2b}+h, -a+k)$
$(\frac{3\pi}{2b}+h, -a+k)$
period $\frac{2\pi}{b}$

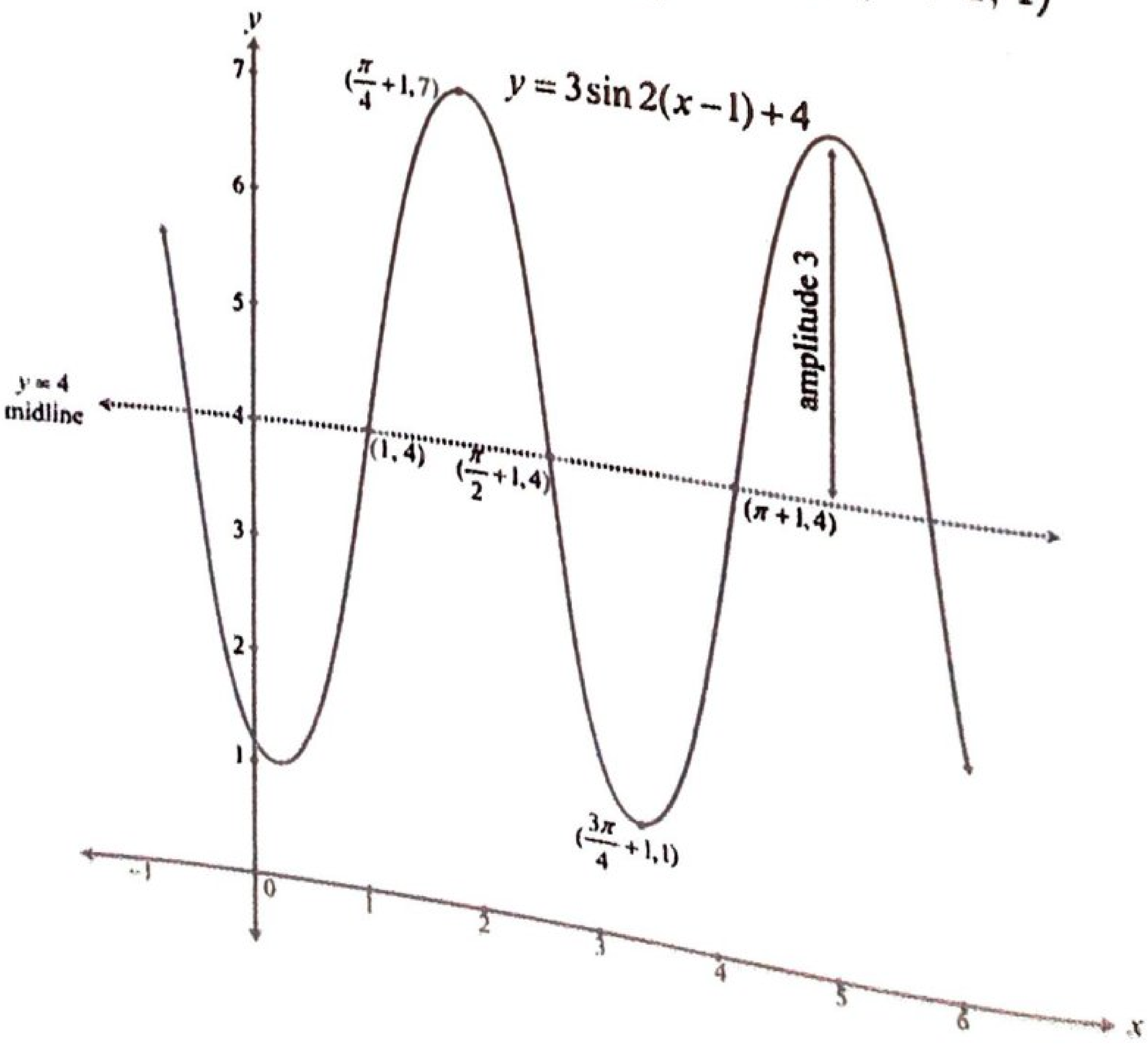
Example 3.

Draw the graph of $y = 3\sin 2(x-1)+4$.

Solution

Midline: $y = 4$, Amplitude: 3, Period: π

Five key points: $(1,4)$, $(\frac{\pi}{4}+1, 7)$, $(\frac{\pi}{2}+1, 4)$, $(\frac{3\pi}{4}+1, 1)$, $(\pi+1, 4)$



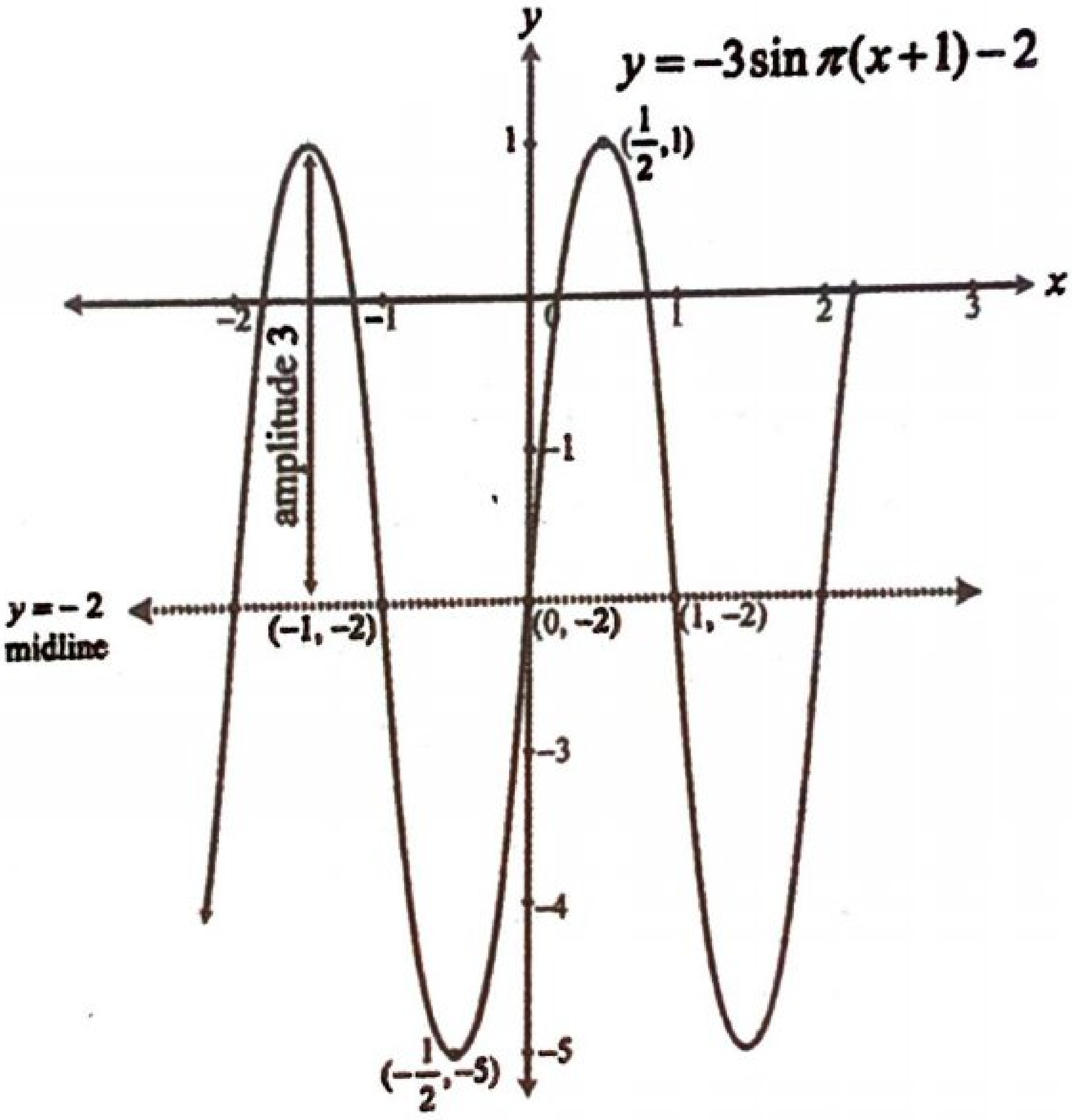
Example 4.

Draw the graph of $y = -3\sin\pi(x+1) - 2$.

Solution

Midline: $y = -2$, Amplitude: 3, Period: 2

Five key points: $(-1,-2)$, $(-\frac{1}{2},-5)$, $(0,-2)$, $(\frac{1}{2},1)$ $(1,-2)$

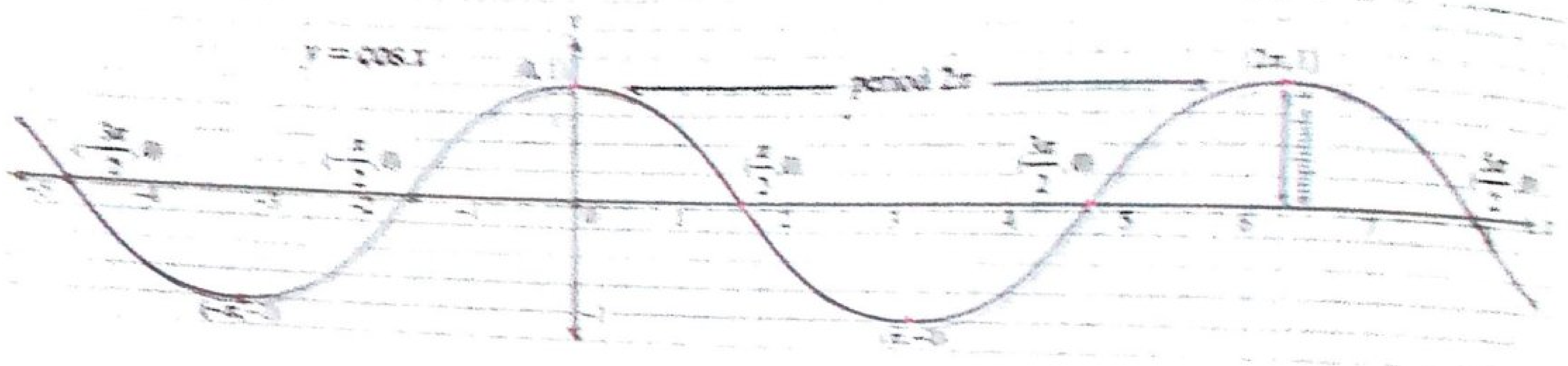


Exercise 7.1

1. From the graph of $y = \sin x$, draw step-by-step transformation graphs to get the graph of $y = -3\sin\frac{\pi}{3}x$.
2. Draw the graph of (a) $y = \frac{1}{2}\sin x$ (b) $y = \sin 4x$.
3. Draw the graph of (a) $y = 2\sin\frac{\pi}{4}x$ (b) $y = -\sin\pi x$.
4. Draw the graph of (a) $y = 2\sin\frac{\pi}{3}(x-2) + 1$ (b) $y = -2\sin\frac{1}{2}(x+1) + 2$.
5. Show that $y = a\sin bx$ is an odd function.

7.2 Graphs of Cosine Functions

Graph of the Cosine Function $y = \cos x$



Domain: The set R of all real numbers Range: $\{y \mid -1 \leq y \leq 1\}$

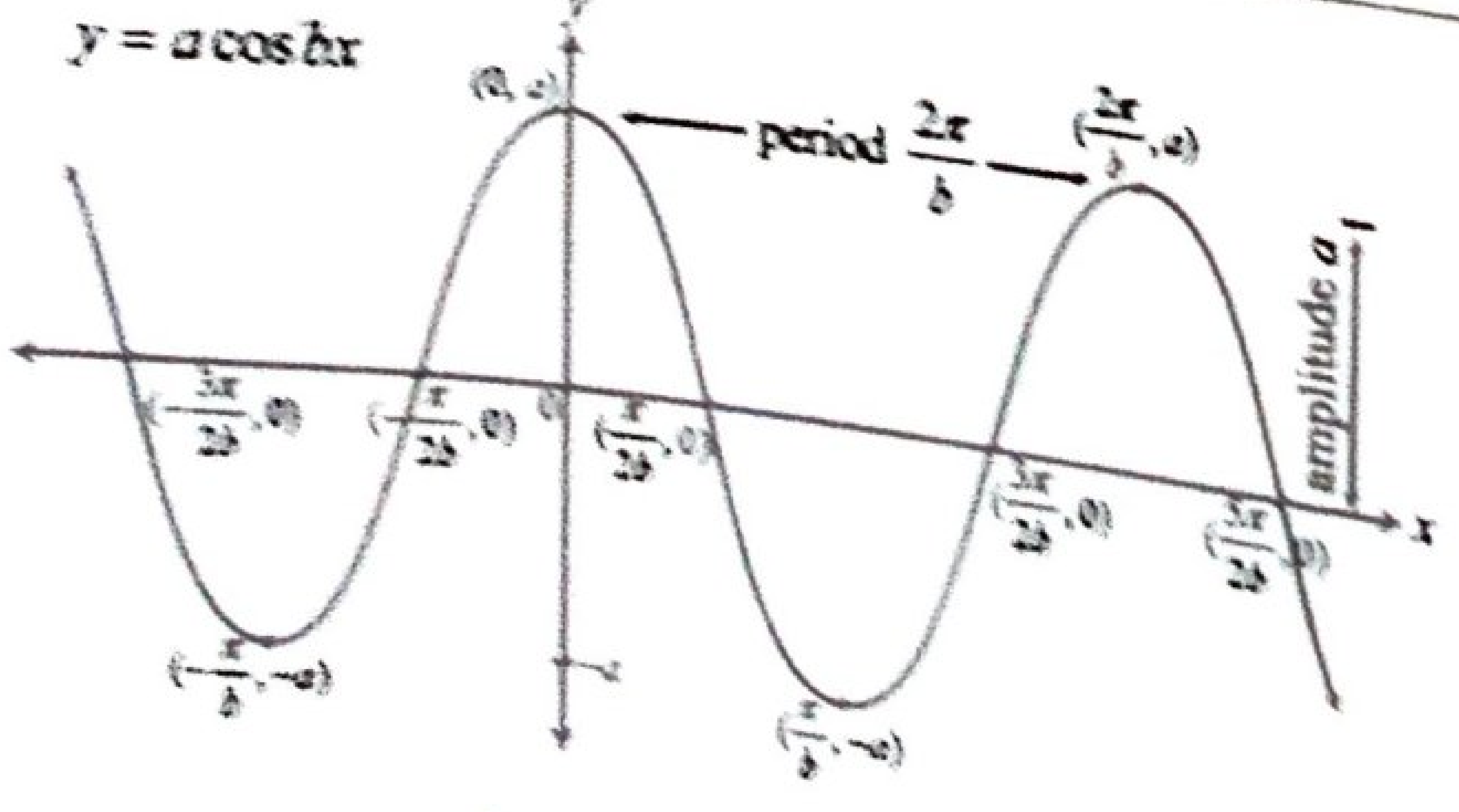
Five key points : $\begin{cases} x\text{-intercepts: } (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0) \\ \text{maximum point: } (0, 1), (2\pi, 1) \end{cases}$ minimum point: $(\pi, -1)$

Note that $\sin(x + \frac{\pi}{2}) = \cos x$, $y = \sin x \xrightarrow[-\frac{\pi}{2} \text{ units}]{\text{horizontal translation}} y = \cos x$

Graph of the Cosine Function $y = a\cos bx$, $a > 0, b > 0$

From the graph of the $y = \cos x$, the graph of $y = a\cos bx$ can be obtained as

$$y = \cos x \xrightarrow[\text{horizontal scaling, scale factor } \frac{1}{b}]{\text{vertical scaling, scale factor } a} y = a\cos bx$$
$$(x, y) \to (\frac{x}{b}, ay)$$



Domain: R

Range: $\{y \mid -a \leq y \leq a\}$

Period: $\frac{2\pi}{b}$

Amplitude: a

Five key points : $\begin{cases} x\text{-intercepts: } (\frac{\pi}{2b}, 0), (\frac{3\pi}{2b}, 0) \\ \text{maximum point: } (0, a), (\frac{2\pi}{b}, a) \end{cases}$ minimum point: $(\frac{\pi}{b}, -a)$

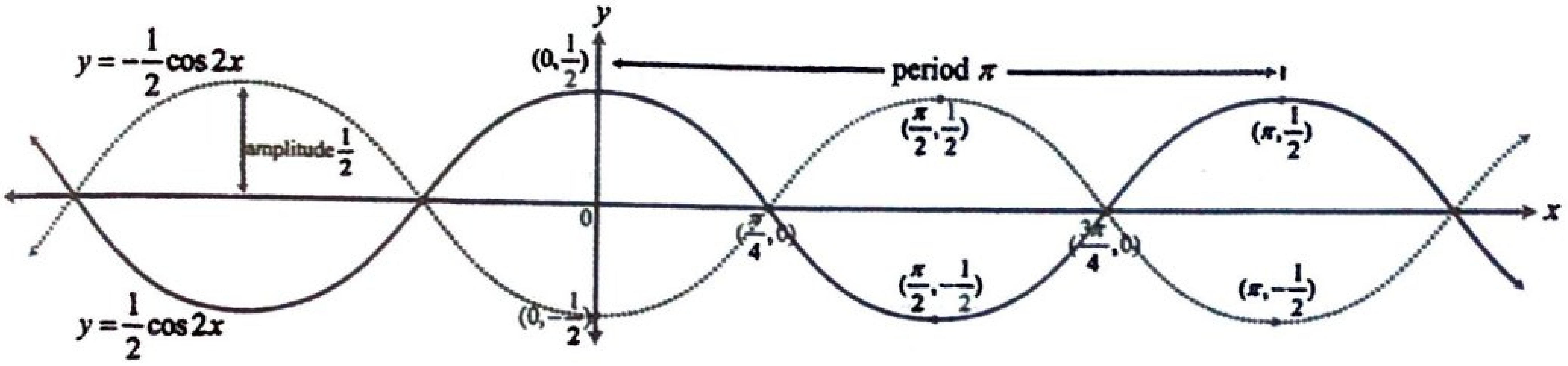
Graph of the Cosine Function $y = -a\cos bx,\ a > 0, b > 0$

$$y = a\cos bx \xrightarrow{\text{Reflection on the } x\text{-axis}} y = -a\cos bx$$

Example 5.

Draw the graphs of $y = \dfrac{1}{2}\cos 2x$ and $y = -\dfrac{1}{2}\cos 2x$.

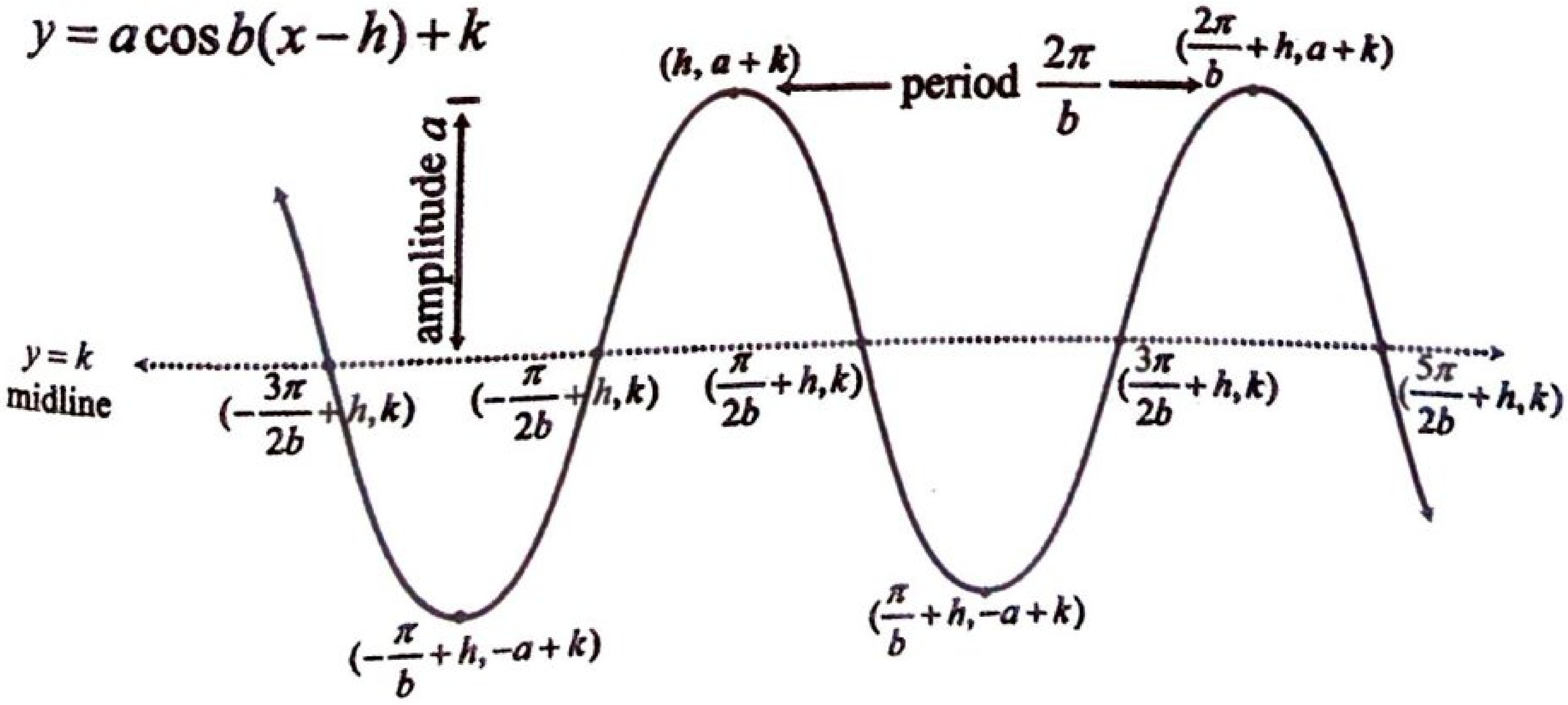
Solution



Note that $y = a\cos(-b)x = a\cos bx$, so no need to consider $b < 0$.

Graph of the Cosine Function $y = a\cos b(x-h) + k,\ a > 0, b > 0$

$$y = = a\cos bx \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} y = a\cos b(x-h) + k$$



$$y = \cos x \rightarrow y = a\cos bx \rightarrow y = a\cos b(x-h) + k$$

$$(x, y) \rightarrow (\frac{x}{b}, ay) \rightarrow (\frac{x}{b} + h, ay + k)$$

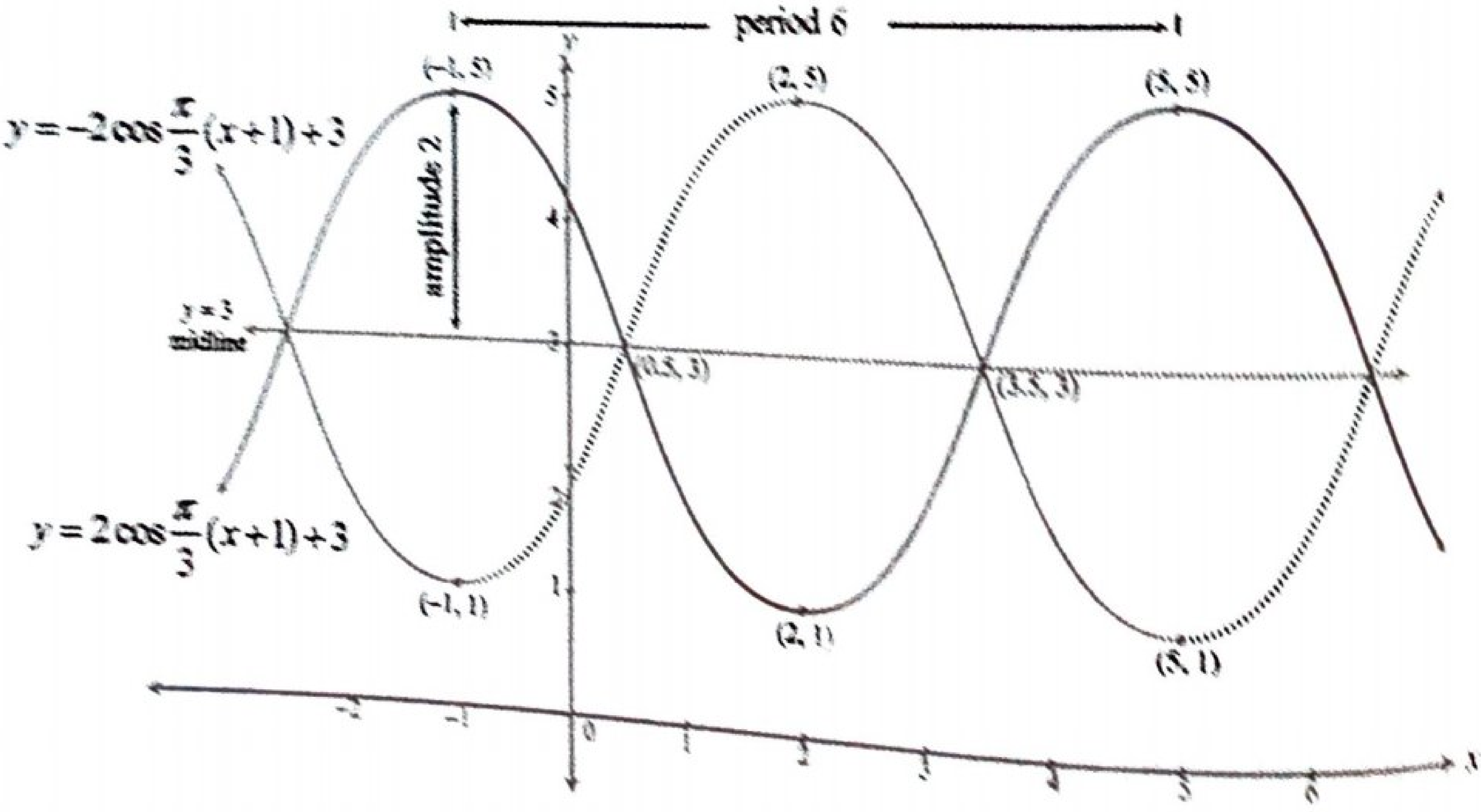
Five key points: points on the midline: $(\frac{\pi}{2b}+h,k)$, $(\frac{3\pi}{2b}+h,k)$
maximum points: $(h,a+k)$, $(\frac{2\pi}{b}+h,a+k)$ minimum point: $(\frac{\pi}{b}+h,-a+k)$

Example 6.

Draw the graphs of $y=2\cos\frac{\pi}{3}(x+1)+3$ and $y=-2\cos\frac{\pi}{3}(x+1)+3$.

Solution

Midline: $y=3$, Amplitude: 2, Period: 6
Five key points for $y=2\cos\frac{\pi}{3}(x+1)+3$: $(-1,5)$, $(0.5,3)$, $(2,1)$, $(3.5,3)$, $(5,5)$
Five key points for $y=-2\cos\frac{\pi}{3}(x+1)+3$: $(-1,1)$, $(0.5,3)$, $(2,5)$, $(3.5,3)$, $(5,1)$

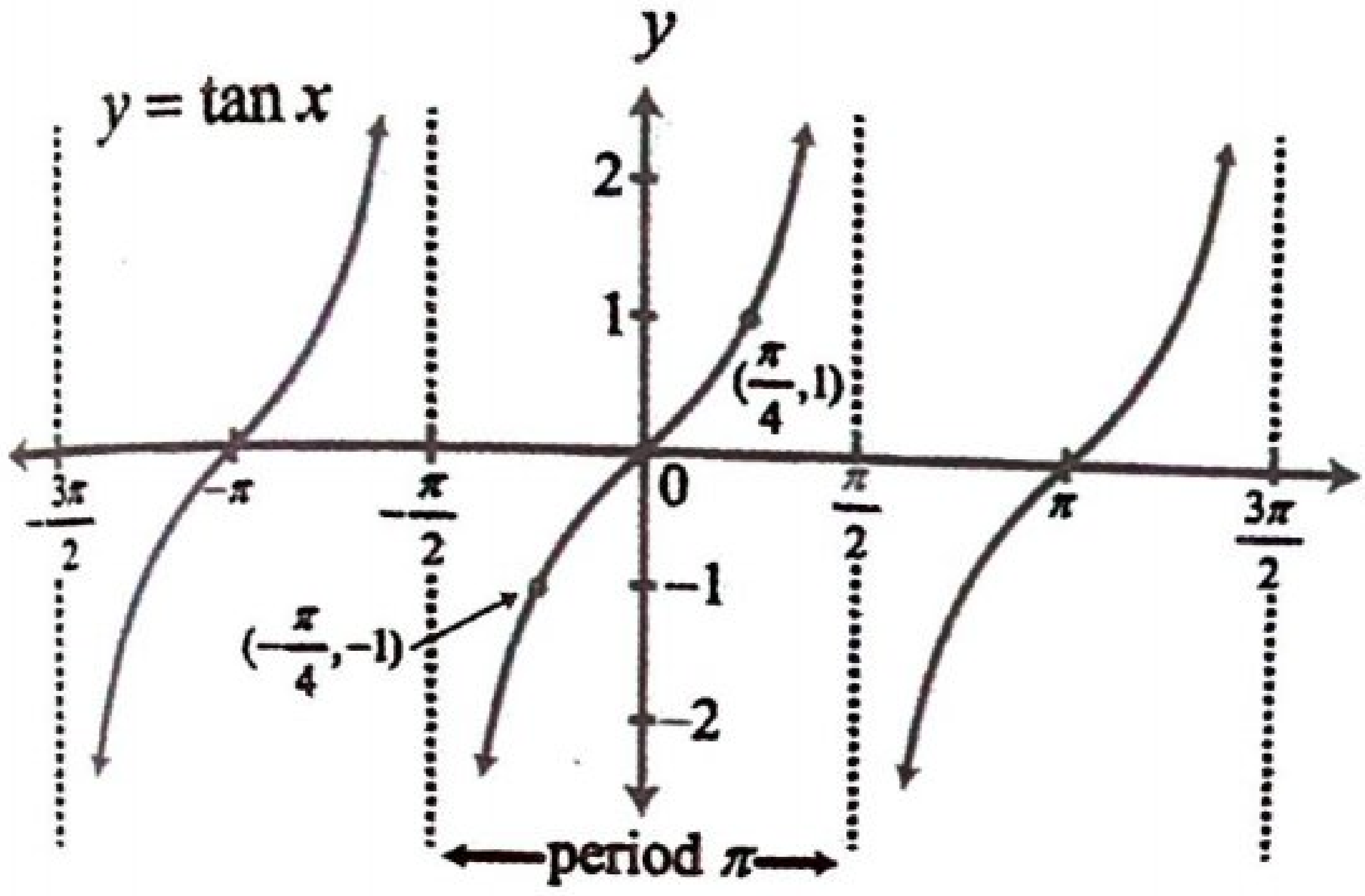


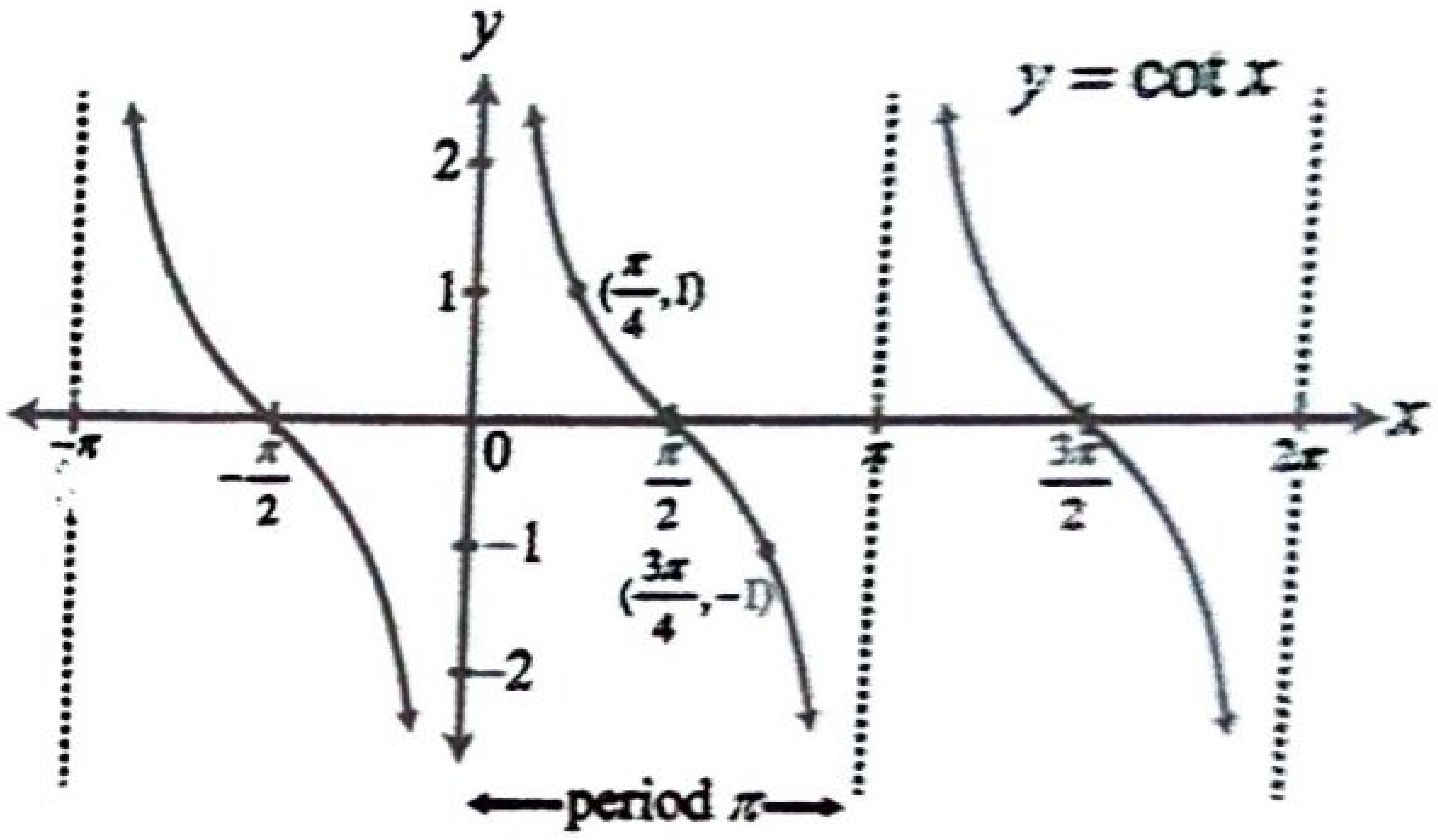
Exercise 7.2

1. From the graph of $y=\cos x$, draw step-by-step transformation graphs to get the graph of $y=-3\cos\frac{\pi}{2}x$.
2. Draw the graph of (a) $y=\frac{1}{2}\cos x$ (b) $y=\cos 4x$.
3. Draw the graph of (a) $y=2\cos\frac{\pi}{4}x$ (b) $y=-\cos\pi x$.
4. Draw the graph of (a) $y=2\cos\pi(x-2)+1$ (b) $y=-2\cos\frac{1}{2}(x+1)+2$.
5. Show that $y=a\cos bx$ is an even function.

7.3 Graphs of Other Trigonometric Functions

Graphs of $y = \tan x$ and $y = \cot x$





Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $\mathbb{R}$
Period: π
x-intercepts: $x = 0, \pm\pi, \pm 2\pi, \ldots$
Asymptotes: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

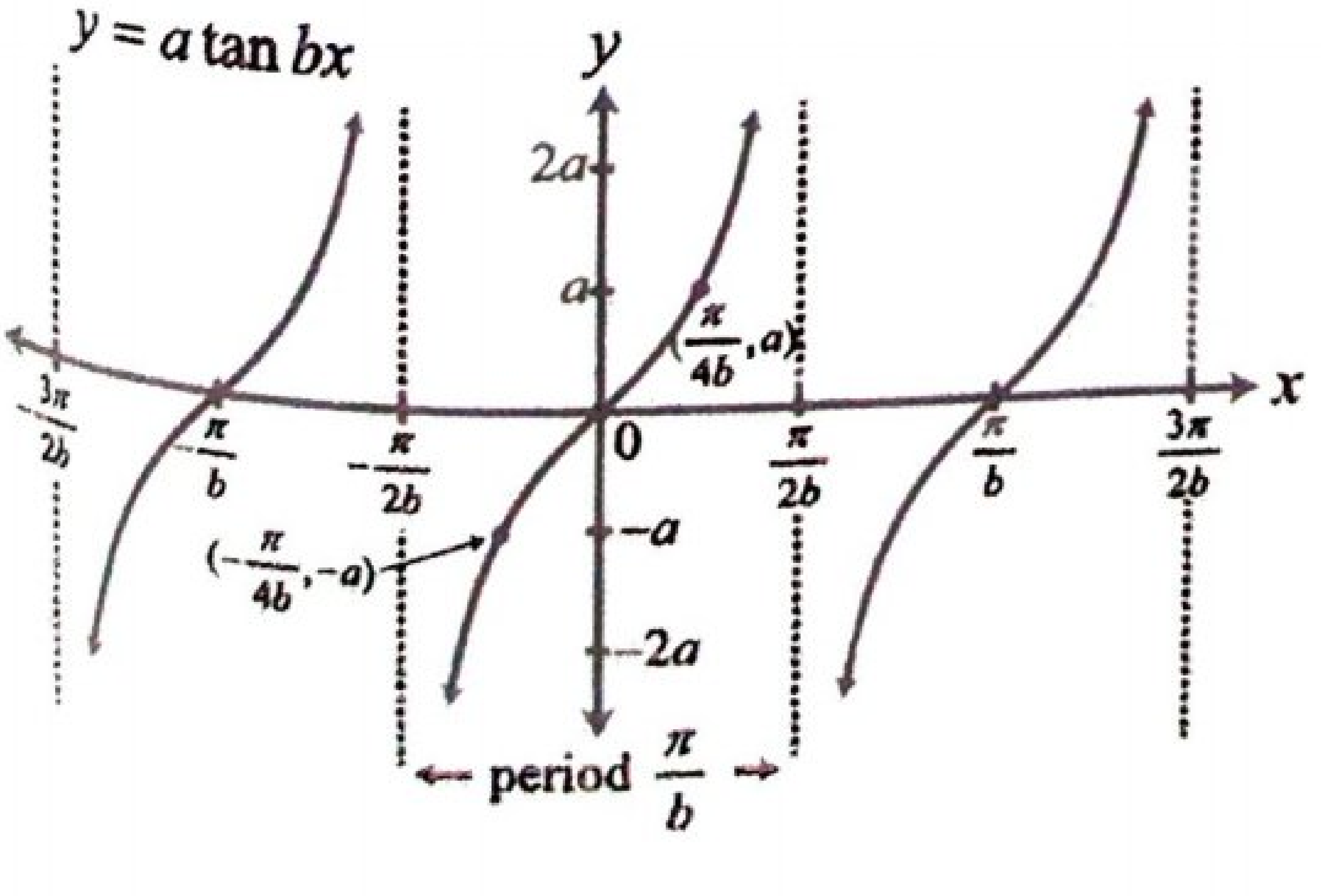
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $\mathbb{R}$
Period: π
x-intercepts: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Asymptotes: $x = 0, \pm\pi, \pm 2\pi, \ldots$

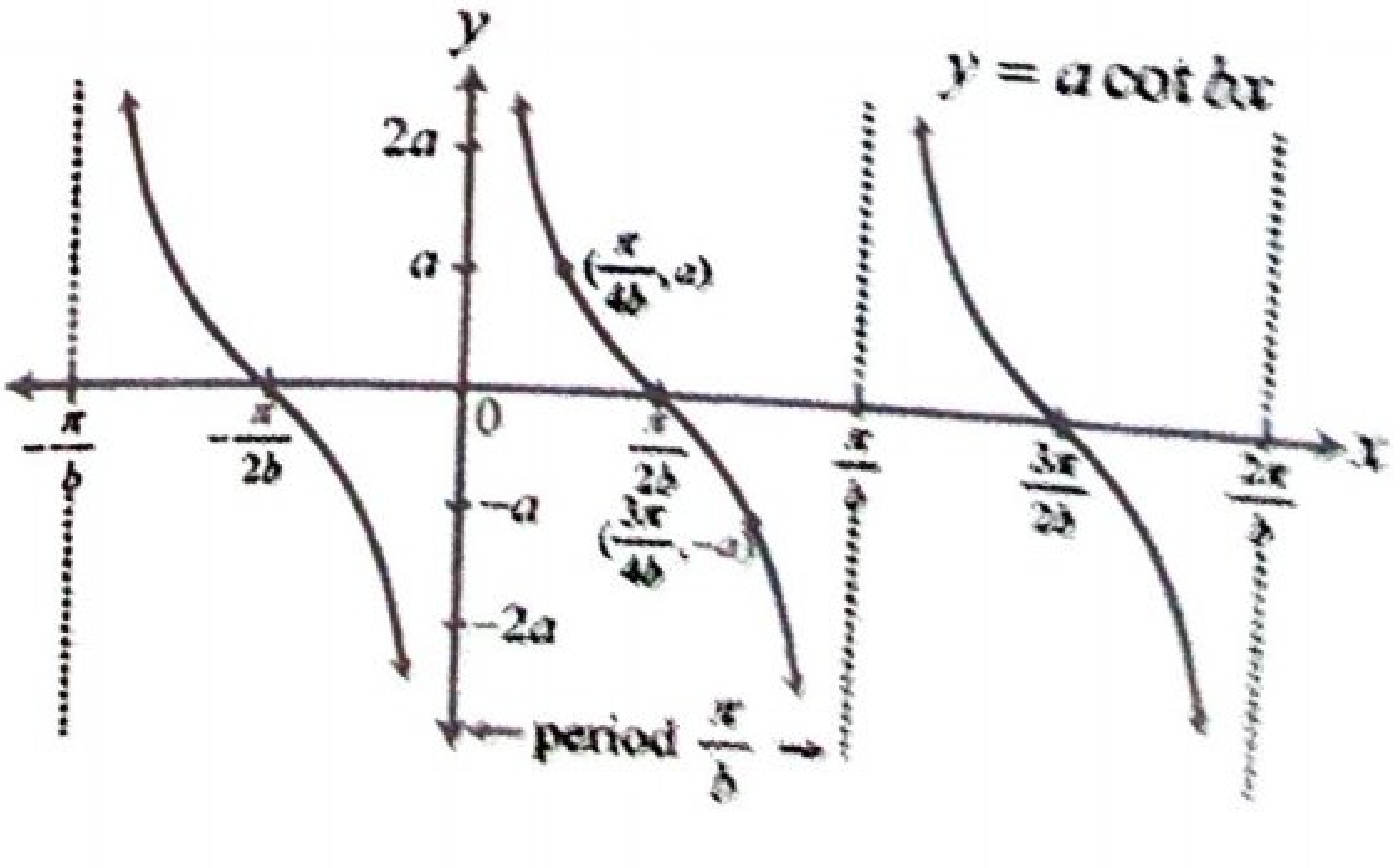
Example 7.

Draw the graphs of $y = a\tan bx$ and $y = a\cot bx$ for $a, b > 0$.

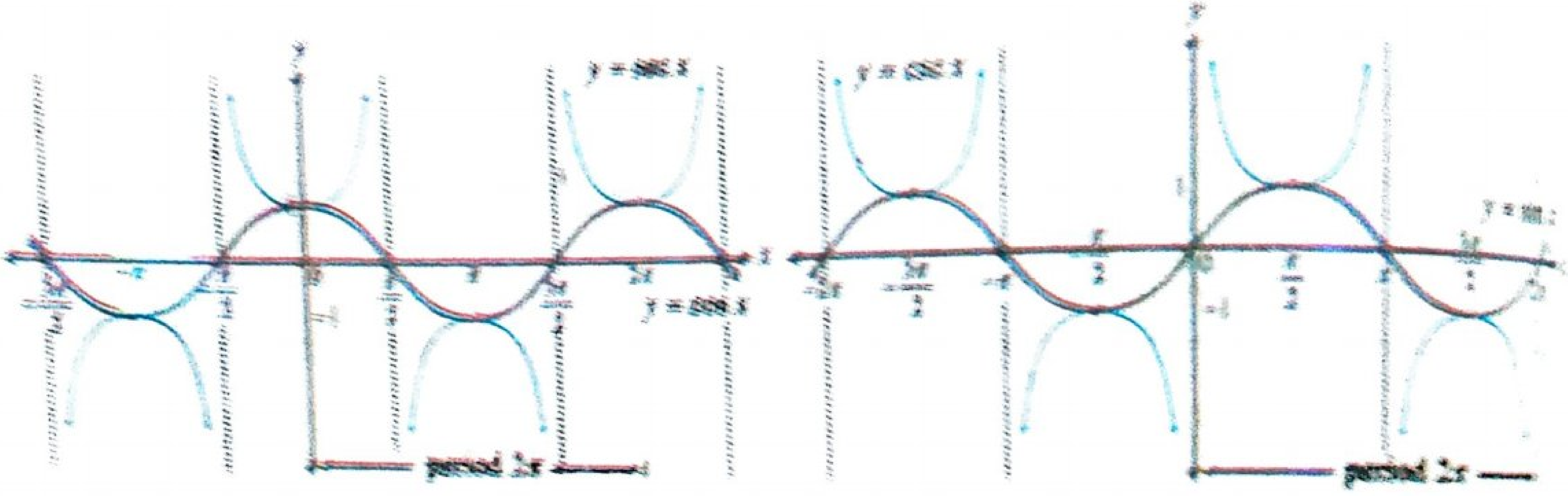
Solution

$$y = \tan x \to y = a\tan bx, \qquad y = \cot x \to y = a\cot bx$$
$$(x, y) \to (\frac{x}{b}, ay)$$





Graphs of $y = \sec x$ and $y = \csc x$



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \leq -1$ or $y \geq 1$
Period: 2π
Asymptotes: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Domain: $x \neq 0, \pm\pi, \pm2\pi, \ldots$
Range: $y \leq -1$ or $y \geq 1$
Period: 2π
Asymptotes: $x = 0, \pm\pi, \pm2\pi, \ldots$

Note that transform $y = \sec x$ to $y = a\sec bx$ and $y = \csc x$ to $y = a\csc bx$ with $(x, y) \to (\frac{x}{b}, ay)$, we get the graphs of $y = a\sec bx$ and $y = a\csc bx$.

Exercise 7.3

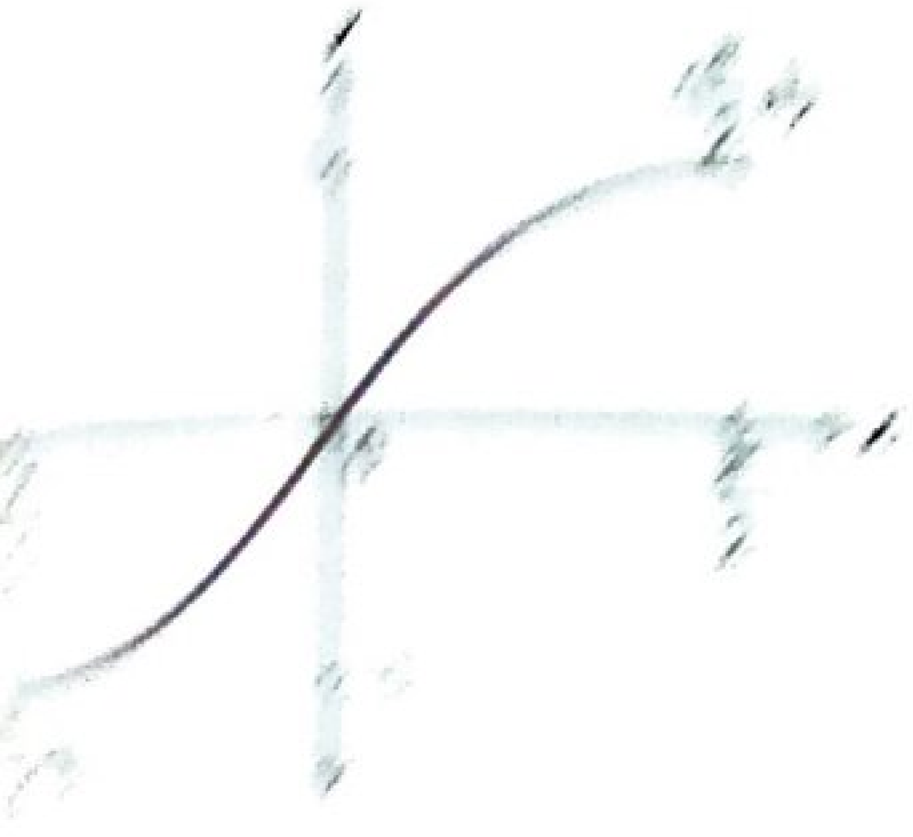
1. Draw the graph of (a) $y = \frac{1}{2}\tan \pi x$ (b) $y = 2\tan\frac{1}{2}x$.
2. Draw the graph of (a) $y = 2\cot\frac{\pi}{3}x$ (b) $y = 3\cot 2x$.
3. Draw the graph of (a) $y = 2\sec \pi x$ (b) $y = 2\csc\frac{1}{2}x$.
4. Show that $y = a\tan bx$, $y = a\cot bx$, and $y = a\csc bx$ are odd functions.
5. Show that $y = a\sec bx$ is an even function.

7.4 Inverse Trigonometric Functions

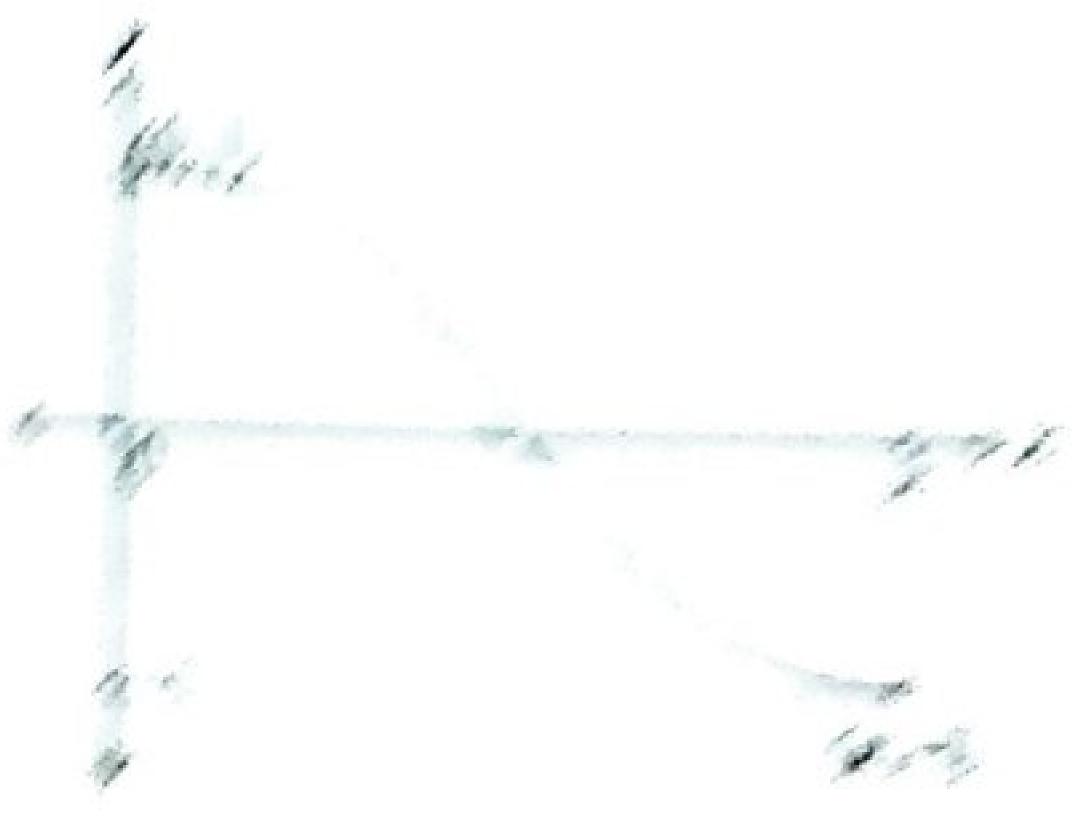
Trigonometric functions are not one-to-one functions. So we need to restrict the domain of each function to be a one-to-one function to get the inverse trigonometric functions. The following graphs show that $y = \sin x$, $y = \cos x$, and $y = \tan x$ are one-to-one in the restricted domains.

In these restricted domains, we can define the inverse functions $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$.

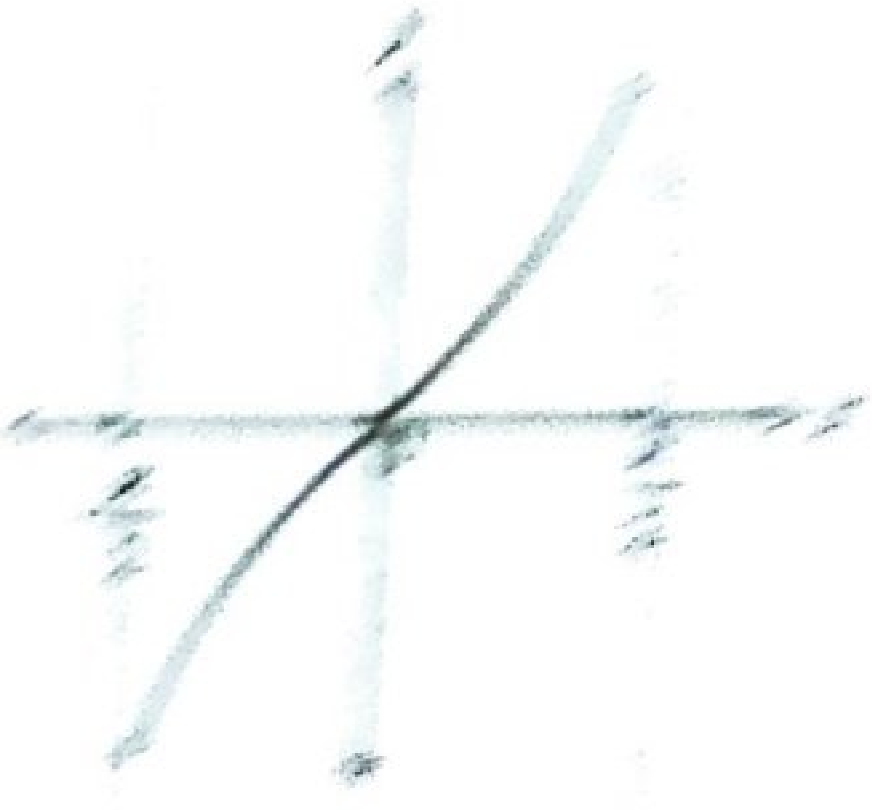
$y = \sin x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$y = \cos x, \ 0 \le x \le \pi$

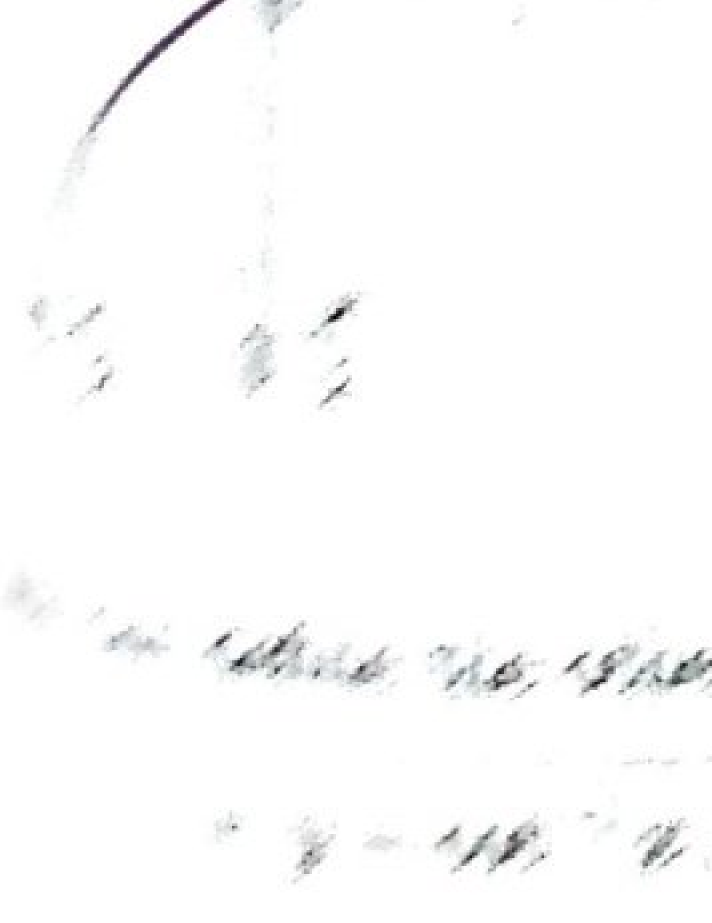


$y = \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

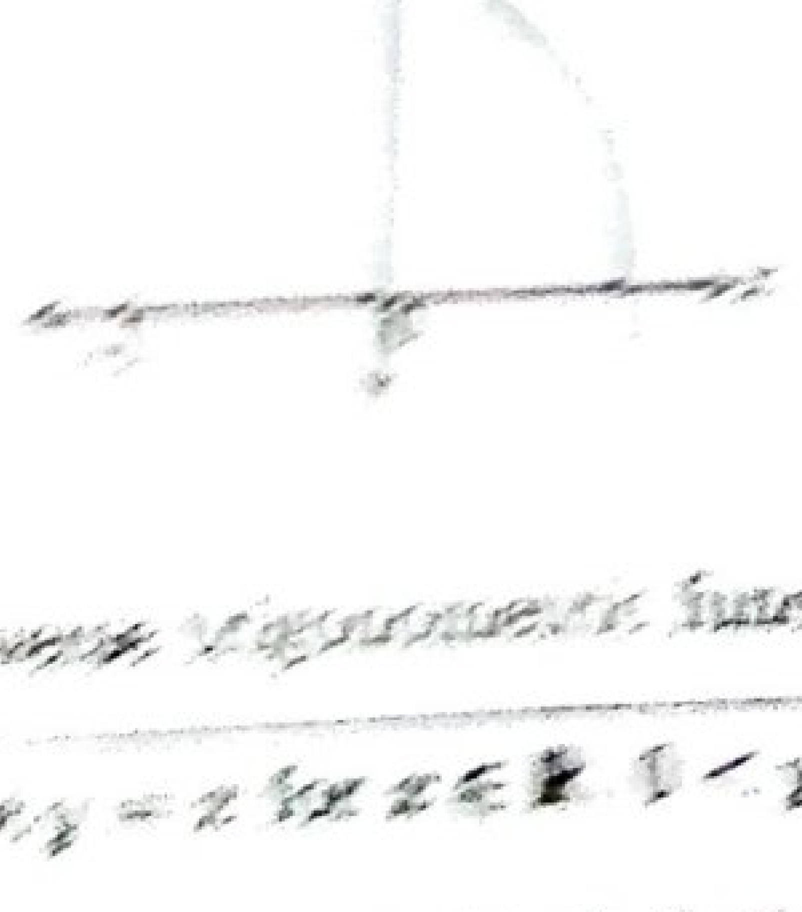


- $y = \sin^{-1} x \Leftrightarrow \sin y = x$ for $-1 \le x \le 1, \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$
- $y = \cos^{-1} x \Leftrightarrow \cos y = x$ for $-1 \le x \le 1, \ 0 \le y \le \pi$
- $y = \tan^{-1} x \Leftrightarrow \tan y = x$ for $x \in \mathbb{R}, \ -\frac{\pi}{2} < y < \frac{\pi}{2}$

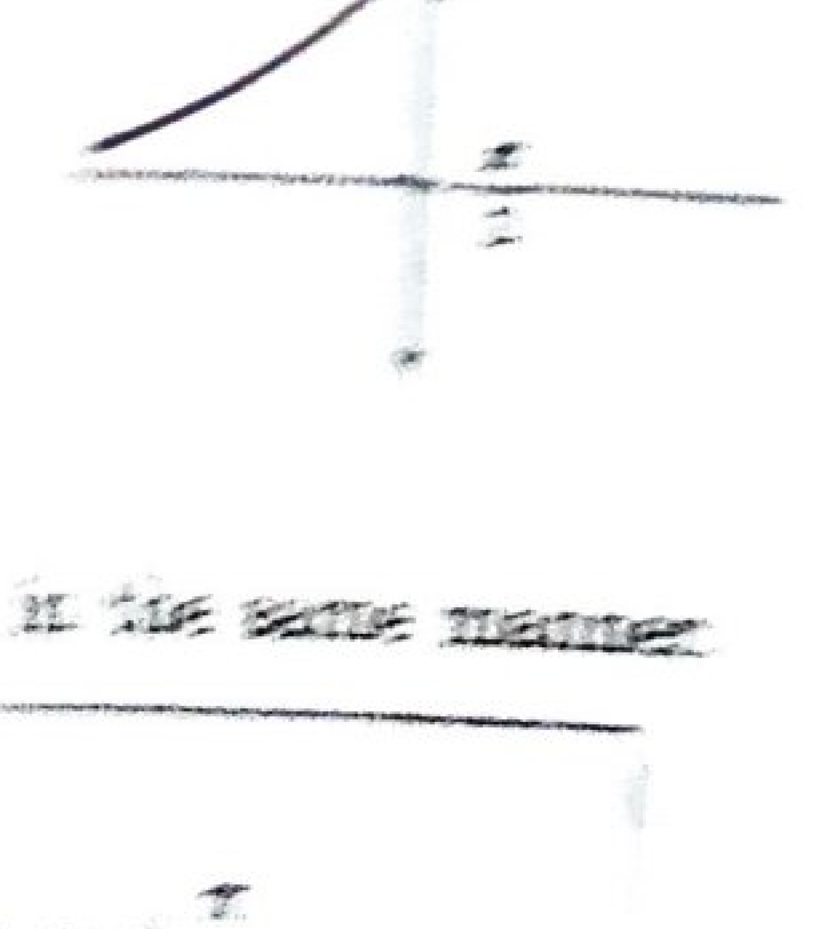
$y = \sin^{-1} x, \ -1 \le x \le 1$



$y = \cos^{-1} x, \ -1 \le x \le 1$



$y = \tan^{-1} x, \ x \in \mathbb{R}$



We define the other inverse trigonometric functions in the same manner.

- $y = \cot^{-1} x \Leftrightarrow \cot y = x$ for $x \in \mathbb{R}, \ 0 < y < \pi$
- $y = \sec^{-1} x \Leftrightarrow \sec y = x$ for $|x| \ge 1, \ 0 \le y \le \pi, \ y \ne \frac{\pi}{2}$
- $y = \csc^{-1} x \Leftrightarrow \csc y = x$ for $|x| \ge 1, \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \ y \ne 0$

Example 8.

Evaluate (a) $\sin^{-1}\frac{1}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(-\sqrt{3})$ (d) $\sin^{-1}(\sin\frac{5\pi}{6})$.

Solution

(a) Since $\sin\frac{\pi}{6} = \frac{1}{2}$ and $-\frac{\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}$, we have $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

(b) Since $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $0 \le \frac{2\pi}{3} \le \pi$, we have $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.

(c) Since $\tan(-\frac{\pi}{3}) = -\sqrt{3}$ and $-\frac{\pi}{2} \le -\frac{\pi}{3} \le \frac{\pi}{2}$, we have $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

(d) $\sin^{-1}(\sin\frac{5\pi}{6}) = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

From $y = \sin^{-1}x$ if $\sin y = x$, we have $\boxed{\sin(\sin^{-1}x) = x}$ for the domain $-1 \le x \le 1$ of $y = \sin^{-1}x$, but from Example 8(d) one can see that $\sin^{-1}(\sin x) =$ not true in general. Since $\sin(-\sin^{-1}x) = -\sin(\sin^{-1}x) = -x$, we have

$$\boxed{\sin^{-1}(-x) = -\sin^{-1}x}$$

so $y = \sin^{-1}x$ is an odd function.

From $y = \cos^{-1}x$ if $\cos y = x$, we have $\boxed{\cos(\cos^{-1}x) = x}$ for the domain $-1 \le x \le 1$ of $y = \cos^{-1}x$. Since $\cos(\pi - \cos^{-1}x) = -\cos(\cos^{-1}x) = -x$,

$$\boxed{\pi - \cos^{-1}x = \cos^{-1}(-x)} \quad \text{or} \quad \boxed{\cos^{-1}x + \cos^{-1}(-x) = \pi}$$

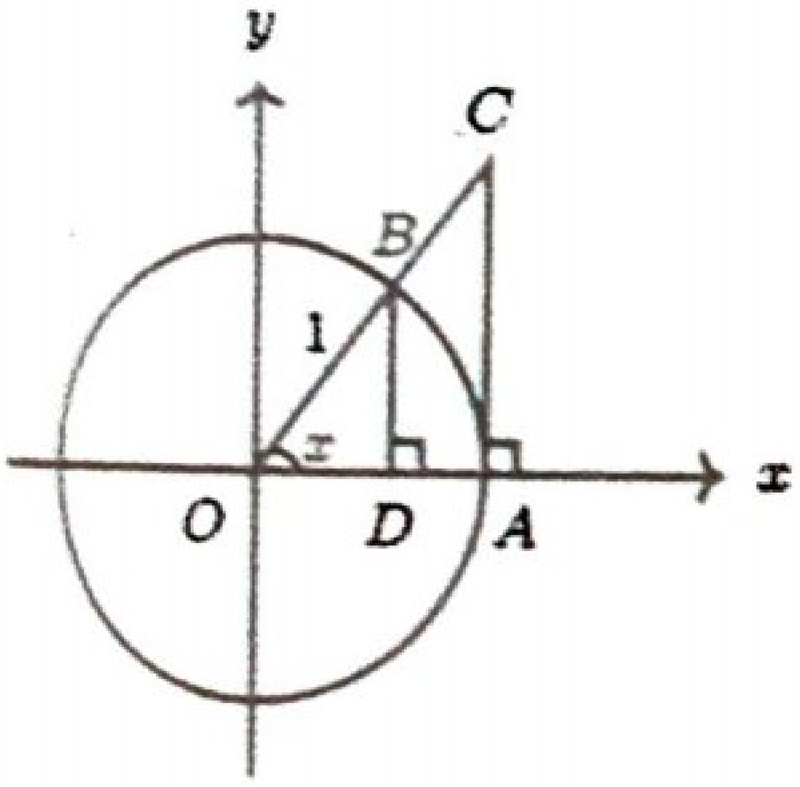
Exercise 7.4

1. Evaluate (a) $\csc(\sin^{-1}0.3)$ (b) $\cot(\tan^{-1}5)$ (c) $\sec(\cos^{-1}(-0.75))$.
2. Evaluate (a) $\sin^{-1}\frac{\sqrt{3}}{2}$ (b) $\cos^{-1}\frac{\sqrt{2}}{2}$ (c) $\tan^{-1}\frac{\sqrt{3}}{3}$.
3. Evaluate (a) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$ (b) $\tan(\sin^{-1}\frac{\sqrt{3}}{2})$ (c) $\csc(\tan^{-1}(-\frac{\sqrt{3}}{3}))$.
4. Evaluate (a) $\sin^{-1}(\sin\frac{2\pi}{3})$ (b) $\cos^{-1}(\cos(-\frac{1}{2}))$ (c) $\tan^{-1}(\tan(-\sqrt{3}))$.
5. Prove that $\sin^{-1}x + \cos^{-1}x = \pi$

7.5 Differentiation of Trigonometric Functions

Before we study the differentiation of trigonometric functions, we first evaluate an important limit, $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

Consider the unit circle in the following figure with radii OA and OB with $OA = OB = 1$ and $\angle AOB = x$ radians. Obviously, $BD <$ arc $AB < AC$. In right $\triangle OBD$, $BD = OB \sin x = \sin x$.



In right $\triangle OAC$, $AC = OA \tan x = \tan x$.
Length of arc $AB = OB \cdot x = x$, we have

$$\sin x < x < \tan x$$

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

When $x \to 0$, $\cos x \to 1$.

Since $\lim\limits_{x \to 0} 1 = 1$ and $\lim\limits_{x \to 0} \cos x = 1$, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

Derivative of $\sin x$

Let

$$y = \sin x.$$

$$y + \delta y = \sin(x + \delta x)$$

$$\therefore \delta y = \sin(x + \delta x) - \sin x$$

$$= 2\cos(x + \frac{\delta x}{2}) \cdot \sin\frac{\delta x}{2} \qquad (\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2})$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos(x + \frac{\delta x}{2}) \cdot \sin\frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos(x + \frac{\delta x}{2}) \cdot \lim_{\delta x \to 0} \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \cos x \times 1 \qquad (\text{when } \delta x \to 0, \frac{\delta x}{2} \to 0)$$

$$\frac{d}{dx} \sin x = \cos x$$

In general, $\frac{d}{dx}\sin u(x) = \cos u(x) \cdot \frac{d}{dx}u(x)$.

Derivative of $\cos x$

Since $\cos x = \sin(\frac{\pi}{2} + x)$

$$\begin{aligned}
\frac{d}{dx}\cos x &= \frac{d}{dx}\sin(\frac{\pi}{2} + x) \\
&= \cos(\frac{\pi}{2} + x) \cdot \frac{d}{dx}(\frac{\pi}{2} + x) \\
&= -\sin x \times 1, \qquad (\because \cos(\frac{\pi}{2} + x) = -\sin x) \\
\frac{d}{dx}\cos x &= -\sin x
\end{aligned}$$

In general, $\frac{d}{dx}\cos u(x) = -\sin u(x) \cdot \frac{d}{dx}u(x)$.

Derivative of $\tan x$

Since $\tan x = \frac{\sin x}{\cos x}$

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\
&= \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}\cos x}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \qquad \text{(quotient formula)} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad (\cos^2 x + \sin^2 x = 1) \\
&= \sec^2 x \qquad (\sec x = \frac{1}{\cos x}) \\
\therefore \frac{d}{dx}\tan x &= \sec^2 x
\end{aligned}$$

In general, $\frac{d}{dx}\tan u(x) = \sec^2 u(x) \cdot \frac{d}{dx}u(x)$.

Similarly, we can easily find the formulas for the derivatives of $\cot x$, $\sec x$ and csc

Formulas for derivatives of trigonometric functions

1	$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$
2	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$
3	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$
4	$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$
5	$\frac{d}{dx}\sec x = \sec x \cdot \tan x$	$\frac{d}{dx}\sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$
6	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$	$\frac{d}{dx}\csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

Example 9.

Differentiate the following functions with respect to x.

(a) $\sin 5x$ (b) $\cos(7x^2 - 2)$ (c) $\tan(6x + 7)$

(d) $5\sec(3x + 1)$ (e) $\frac{\cot(1 - 2x)}{3}$ (f) $-2\csc 3x$

Solution

(a) $\frac{d}{dx}\sin 5x = \cos 5x \cdot \frac{d}{dx}5x = \cos 5x \cdot 5 = 5 \cdot \cos 5x$

(b) $\frac{d}{dx}\cos(7x^2 - 2) = -\sin(7x^2 - 2) \cdot \frac{d}{dx}(7x^2 - 2) = -\sin(7x^2 - 2) \cdot (14x)$

(c) $\frac{d}{dx}\tan(6x + 7) = \sec^2(6x + 7) \cdot \frac{d}{dx}(6x + 7) = \sec^2(6x + 7) \cdot 6$

(d)
$$\begin{aligned}\frac{d}{dx}5\sec(3x + 1) &= 5 \cdot \sec(3x + 1) \cdot \tan(3x + 1) \cdot \frac{d}{dx}(3x + 1)\\ &= 5 \cdot \sec(3x + 1) \cdot \tan(3x + 1) \cdot 3\\ &= 15 \cdot \sec(3x + 1) \cdot \tan(3x + 1)\end{aligned}$$

(e)
$$\begin{aligned}\frac{d}{dx}\frac{\cot(1 - 2x)}{3} &= -\frac{1}{3} \cdot \csc^2(1 - 2x) \cdot \frac{d}{dx}(1 - 2x)\\ &= -\frac{1}{3}\csc^2(1 - 2x) \cdot (-2)\\ &= \frac{2}{3}\csc^2(1 - 2x)\end{aligned}$$

(f)
$$\begin{aligned}\frac{d}{dx}(-2\csc 3x) &= (-2)(-\csc 3x \cdot \cot 3x) \cdot \frac{d}{dx}3x\\ &= 2\csc 3x \cdot \cot 3x \cdot 3\\ &= 6\csc 3x \cdot \cot 3x\end{aligned}$$

Formulas for derivatives of trigonometric functions

1	$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$
2	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$
3	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$
4	$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$
5	$\frac{d}{dx}\sec x = \sec x \cdot \tan x$	$\frac{d}{dx}\sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$
6	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$	$\frac{d}{dx}\csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

Example 9.

Differentiate the following functions with respect to x.

(a) $\sin 5x$ (b) $\cos(7x^2 - 2)$ (c) $\tan(6x + 7)$

(d) $5\sec(3x + 1)$ (e) $\frac{\cot(1 - 2x)}{3}$ (f) $-2\csc 3x$

Solution

(a) $$\frac{d}{dx}\sin 5x = \cos 5x \cdot \frac{d}{dx}5x = \cos 5x \cdot 5 = 5 \cdot \cos 5x$$

(b) $$\frac{d}{dx}\cos(7x^2 - 2) = -\sin(7x^2 - 2) \cdot \frac{d}{dx}(7x^2 - 2) = -\sin(7x^2 - 2) \cdot (14x)$$

(c) $$\frac{d}{dx}\tan(6x + 7) = \sec^2(6x + 7) \cdot \frac{d}{dx}(6x + 7) = \sec^2(6x + 7) \cdot 6$$

(d) $$\begin{aligned}\frac{d}{dx}5\sec(3x + 1) &= 5 \cdot \sec(3x + 1) \cdot \tan(3x + 1) \cdot \frac{d}{dx}(3x + 1)\\ &= 5 \cdot \sec(3x + 1) \cdot \tan(3x + 1) \cdot 3\\ &= 15 \cdot \sec(3x + 1) \cdot \tan(3x + 1)\end{aligned}$$

(e) $$\begin{aligned}\frac{d}{dx}\frac{\cot(1 - 2x)}{3} &= -\frac{1}{3} \cdot \csc^2(1 - 2x) \cdot \frac{d}{dx}(1 - 2x)\\ &= -\frac{1}{3}\csc^2(1 - 2x) \cdot (-2)\\ &= \frac{2}{3}\csc^2(1 - 2x)\end{aligned}$$

(f) $$\begin{aligned}\frac{d}{dx}(-2\csc 3x) &= (-2)(-\csc 3x \cdot \cot 3x) \cdot \frac{d}{dx}3x\\ &= 2\csc 3x \cdot \cot 3x \cdot 3\\ &= 6\csc 3x \cdot \cot 3x\end{aligned}$$

Example 10.

Find $\frac{dy}{dx}$.

(a) $y = \sin^2 x$ (b) $y = \cos\sqrt{x}$ (c) $y = \tan^2(x^2)$

(d) $y = \sin 2x - x\cos x$ (e) $y = \sin x \cdot \cos^2 x$ (f) $y = \frac{x}{\tan x}$

(g) $y = \sqrt{x + \sin x}$

Solution

(a)
$$\begin{aligned} y &= \sin^2 x \\ \frac{dy}{dx} &= 2\sin x\cos x \end{aligned}$$

(b)
$$\begin{aligned} y &= \cos\sqrt{x} = \cos x^{1/2} \\ \frac{dy}{dx} &= -\sin\sqrt{x}\cdot\frac{1}{2}x^{-1/2} \end{aligned}$$

(c)
$$\begin{aligned} y &= \tan^2(x^2) \\ \frac{dy}{dx} &= 2\cdot\tan(x^2)\cdot\sec^2(x^2)\cdot 2x \end{aligned}$$

(d)
$$\begin{aligned} y &= \sin 2x - x\cos x \\ \frac{dy}{dx} &= (\cos 2x)2 - (x\cdot(-\sin x) + \cos x) \\ &= 2\cos 2x + x\sin x - \cos x \end{aligned}$$

(e)
$$\begin{aligned} y &= \sin x\cdot\cos^2 x \\ \frac{dy}{dx} &= \sin x\cdot\frac{d}{dx}(\cos^2 x) + \cos^2 x\cdot\frac{d}{dx}(\sin x) \\ &= \sin x\cdot 2\cos x\cdot(-\sin x) + \cos^2 x\cdot\cos x \\ &= -2\sin^2 x\cdot\cos x + \cos^3 x \end{aligned}$$

(f)
$$\begin{aligned} y &= \frac{x}{\tan x} \\ \frac{dy}{dx} &= \frac{\tan x - x\cdot\sec^2 x}{\tan^2 x} \end{aligned}$$

(g)
$$\begin{aligned} y &= \sqrt{x + \sin x} = (x + \sin x)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}(x + \sin x)^{-1/2}\cdot(1 + \cos x) \end{aligned}$$

Example 11.

Given that $x + \sin y = \cos(xy)$, find $\dfrac{dy}{dx}$.

Solution

$$x + \sin y =.\cos(xy)$$

Differentiate with respect to x,

$$\begin{aligned}
1 + \cos y \cdot \frac{dy}{dx} &= -\sin(xy) \cdot \frac{d}{dx}(xy) \\
1 + \cos y \cdot \frac{dy}{dx} &= -\sin(xy) \cdot (x \cdot \frac{dy}{dx} + y) \\
(\cos y + x\sin(xy))\frac{dy}{dx} &= -(1 + y \cdot \sin(xy)) \\
\frac{dy}{dx} &= -\frac{1 + y \cdot \sin(xy)}{\cos y + x\sin(xy)}
\end{aligned}$$

Example 12.

Given that $y = x\sin x$, find $\dfrac{d^2y}{dx^2}$.

Solution

$$\begin{aligned}
y &= x \cdot \sin x \\
\frac{dy}{dx} &= x \cdot \cos x + \sin x \\
\frac{d^2y}{dx^2} &= x(-\sin x) + \cos x + \cos x = 2\cos x - x\sin x
\end{aligned}$$

Exercise 7.5

1. Differentiate the following functions with respect to x.

   (a) $\sin(2x + 3)$ (b) $\cos\dfrac{3}{x}$ (c) $x^3\cos 2x$ (d) $\cos 7x + \sin 3x$

   (e) $\sin x \cdot \cos 2x$ (f) $\cos^2(5x)$ (g) $\tan^3\sqrt{x}$ (h) $\sin(\cos x)$

   (i) $\dfrac{\sin x}{1 + \tan x}$ (j) $\sqrt{\sin x + \cos x}$

2. Find $\dfrac{dy}{dx}$.

   (a) $y = \sin(1 - x^2)$ (b) $y = 2\pi x + 2\cos\pi x$ (c) $y = \sin^2 x \cdot \cos 3x$

   (d) $y = x^2\sin(\dfrac{1}{x})$ (e) $3x^2 + 2\sin y = y^2$ (f) $\sin x \cdot \cos y = 2y$

3. Given that $y = \cos^2 x$, prove that $\dfrac{d^2y}{dx^2} + 4y = 2$.

4. Given that $y = \dfrac{1}{3}\cos^3 x - \cos x$, prove that $\dfrac{dy}{dx} = \sin^3 x$.

Chapter [illegible]

Logarithmic and Exponential Functions

[illegible]

[illegible] Logarithmic Function

[illegible]

[illegible]

For $b > 1$, $y = \log_b x$ is an increasing **function** that means if $x_1 > x_2$ then $y_1 = \log_b x_1 > y_2 = \log_b x_2$. For example, $y = \log_2 x$ is an increasing function.

For $0 < b < 1$, $y = \log_b x$ is a decreasing **function** that means if $x_1 > x_2$ then $y_1 = \log_b x_1 < y_2 = \log_b x_2$. For example, $y = \log_{\frac{1}{2}} x$ is a decreasing function.

Since $\log_{\frac{1}{b}} x = -\log_b x$,

$$\boxed{y = \log_b x \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = \log_{\frac{1}{b}} x}$$

as the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$.

For $b > 1$, the graph of $y = \log_b x$ is vertical scaling with scale factor $\dfrac{1}{\log_2 b}$ of the graph $y = \log_2 x$.

$$\boxed{\log_b x = \frac{\log_2 x}{\log_2 b} \quad \text{so that} \quad y = \log_2 x \xrightarrow[\text{scale factor } \frac{1}{\log_2 b}]{\text{vertical scaling}} y = \log_b x}$$

For $0 < b < 1$, the graph of $y = \log_b x$ is vertical scaling with scale factor $\dfrac{1}{\log_{\frac{1}{2}} b}$ of the graph $y = \log_{\frac{1}{2}} x$.

$$\boxed{\log_b x = \frac{\log_{\frac{1}{2}} x}{\log_{\frac{1}{2}} b} \quad \text{so that} \quad y = \log_{\frac{1}{2}} x \xrightarrow[\text{scale factor } \frac{1}{\log_{\frac{1}{2}} b}]{\text{vertical scaling}} y = \log_b x}$$

Therefore all graphs of $y = \log_b x$ are vertical scaling of the graph of $y = \log_2 x$ or the graph of $y = \log_{\frac{1}{2}} x$. So as in $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$, we have

$b > 1$		$0 < b < 1$	
$y = \log_b x < 0$,	if $0 < x < 1$	$y = \log_b x > 0$,	if $0 < x < 1$
$y = \log_b x = 0$,	if $x = 1$	$y = \log_b x = 0$,	if $x = 1$
$y = \log_b x > 0$,	if $x > 1$	$y = \log_b x < 0$,	if $x > 1$

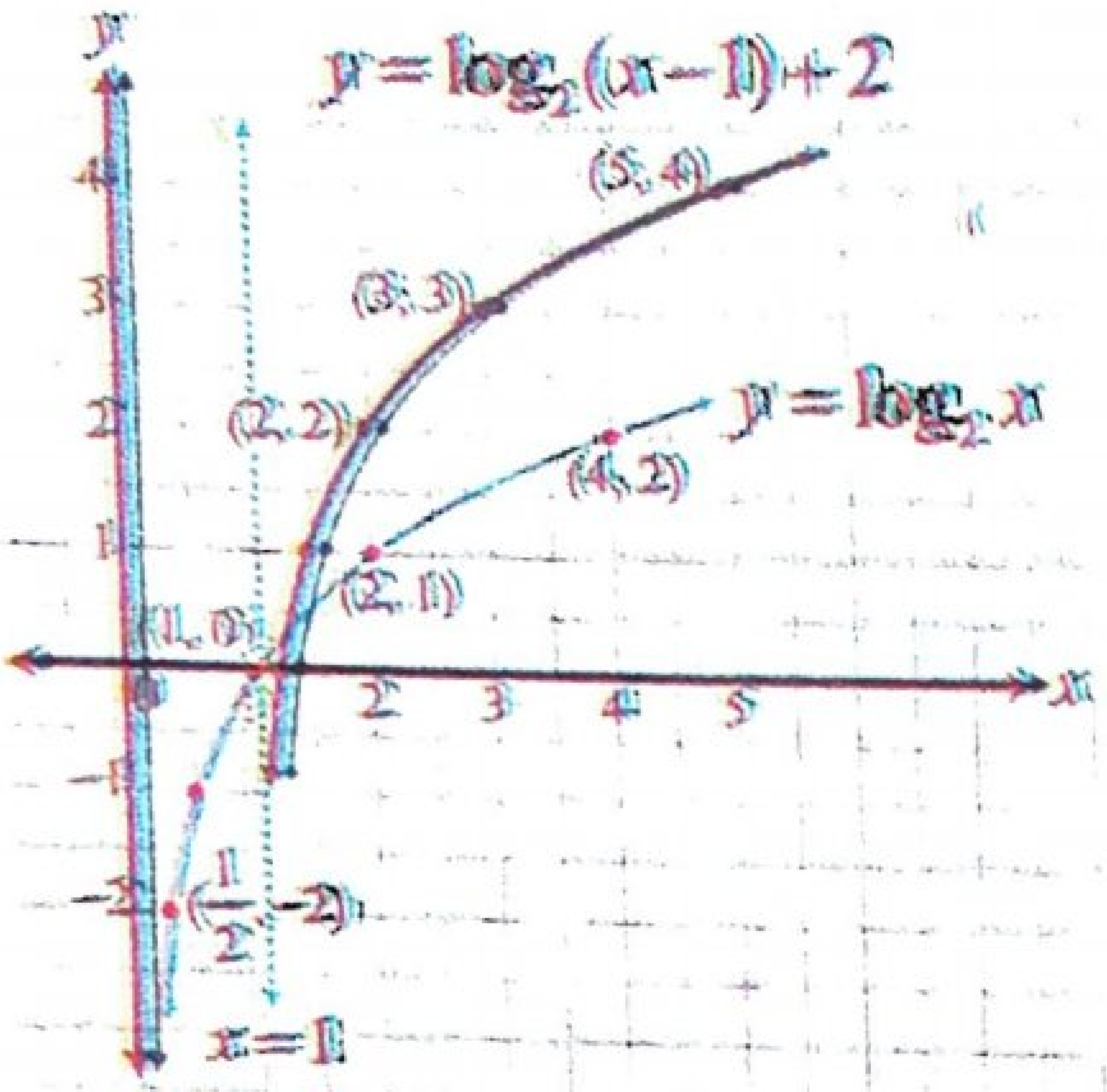
Example 1.

From the graph of $y = \log_2 x$, draw step-by-step transformation graphs to get the graph of (a) $y = \log_2(x-1)+2$ (b) $y = \log_{\frac{1}{2}}(x+2) - 1$.

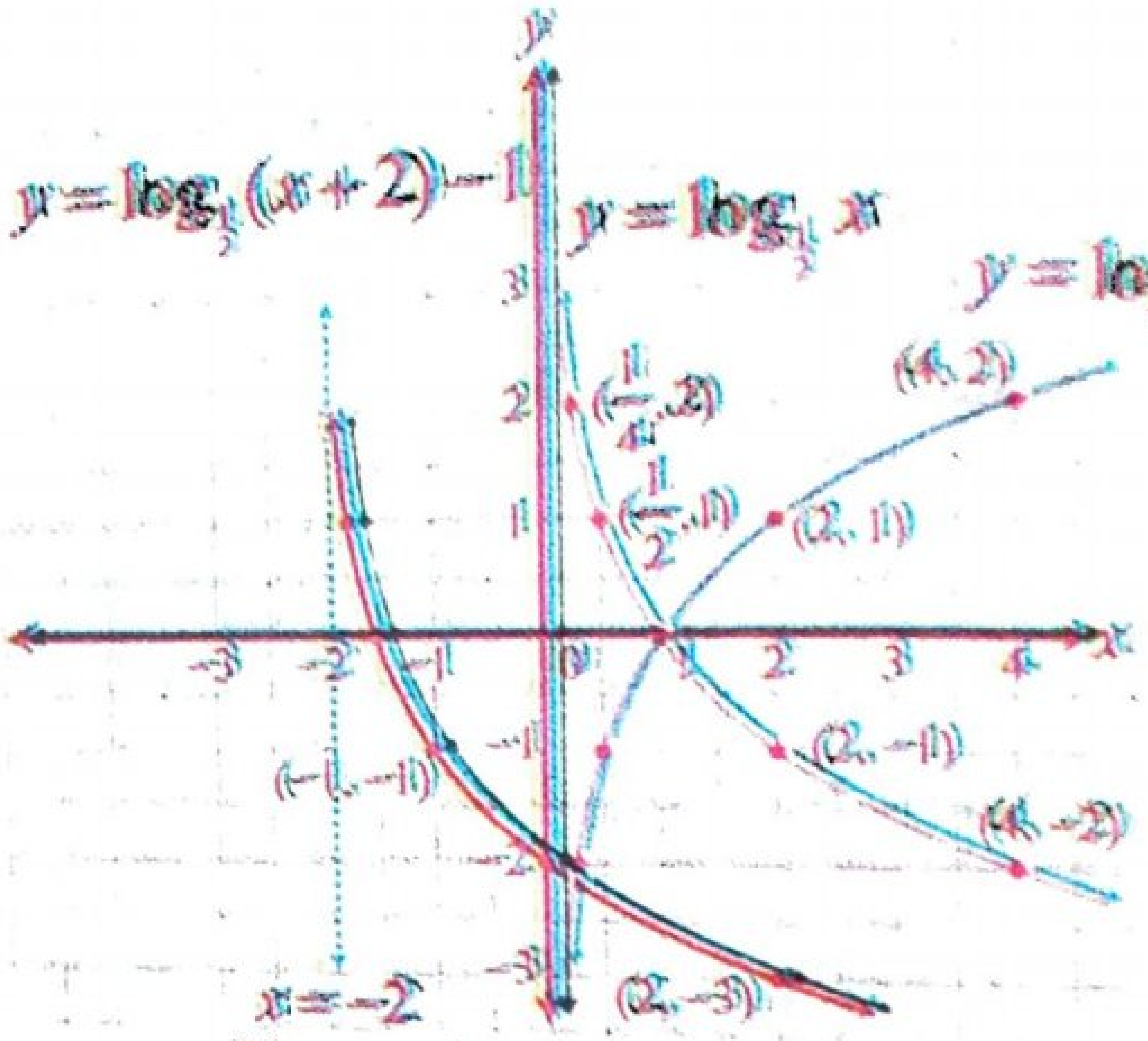
Solution

(a) $$y = \log_2 x \xrightarrow[\text{vertical translation 2 units}]{\text{horisontal translation 1 unit}} y = \log_2(x-1)+2$$

(b) $y = \log_2 x \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = \log_{\frac{1}{2}} x \xrightarrow[\text{vertical translation } -1 \text{ unit}]{\text{horizontal translation } -2 \text{ units}} y = \log_{\frac{1}{2}}(x+2) - 1$



Note: Asymptote $x = 1$



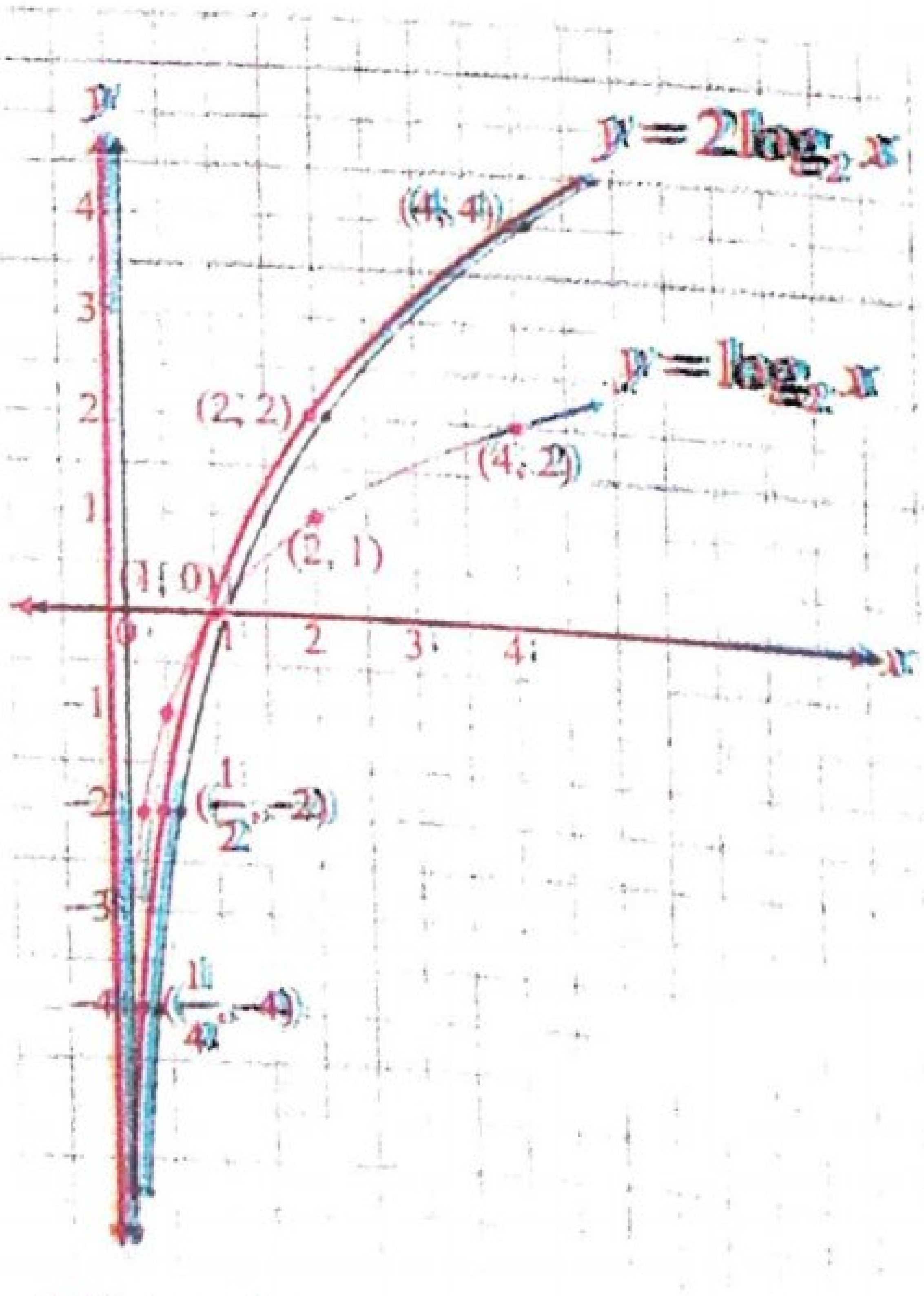
Note: Asymptote $x = -2$

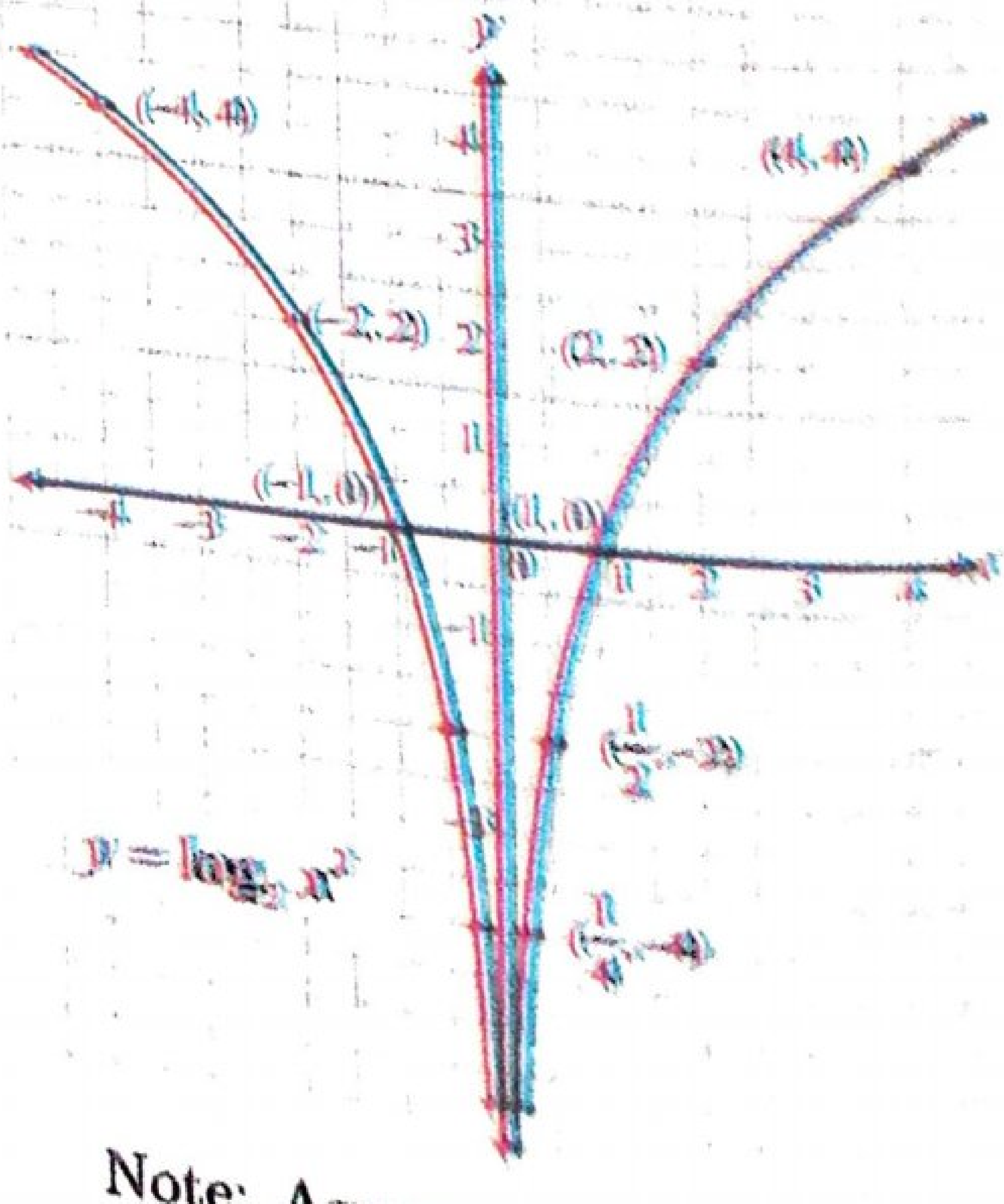
Example 2.

Draw the graphs $y = 2\log_2 x$ and $y = \log_2 x^2$.

Solution

The graph of $y = 2\log_2 x$ can be drawn directly or can be drawn as the vertical scaling with scale factor 2 of the graph of $y = \log_2 x$. Domain of $y = \log_2 x^2$ is $\mathbb{R} \setminus \{0\}$ and $y = \log_2 x^2$ is an even function.





Note: Asymptote y-axis $(x = 0)$

Note: Asymptote y-axis $(x = 0)$

Note that $2\log_2 x = \log_2 x^2$ for $x > 0$.

Exercise 8.1

1. Draw the graph of

   (a) $y = \log_2(x-2)+1$ (b) $y = \log_2(x+1)-2$

   (c) $y = \log_{\frac{1}{2}}(x+1)+2$ (d) $y = \log_{\frac{1}{2}}(x-2)-1$.

2. Draw the graph of (a) $y = \log_{\frac{1}{2}} x^2$ (b) $y = 2\log_{\frac{1}{2}} x$.

3. Draw the graph of (a) $y = \log_2(-x)$ (b) $y = \log_{\frac{1}{2}}(-x)$.

4. Draw the graph of (a) $y = \log_2 |x|$ (b) $y = \log_{\frac{1}{2}} |x|$.

8.2 Differentiation of Logarithmic Functions

Derivative of $y = \log_b x$

Let $y = \log_b x$.
Let δx be a small increment in x and δy be corresponding small increment in y. Then

$$\begin{aligned}
y + \delta y &= \log_b(x+\delta x) \\
\delta y &= \log_b(x+\delta x) - \log_b x \\
&= \log_b \frac{x+\delta x}{x} \\
\frac{\delta y}{\delta x} &= \frac{1}{\delta x} \log_b \frac{x+\delta x}{x} \\
&= \log_b(\frac{x+\delta x}{x})^{1/\delta x} \\
&= \log_b(1+\frac{\delta x}{x})^{1/\delta x}
\end{aligned}$$

Let $\frac{\delta x}{x} = t$, then $\frac{1}{\delta x} = \frac{1}{xt}$.

$$\begin{aligned}
\frac{\delta y}{\delta x} &= \log_b(1+t)^{1/xt} \\
&= \frac{1}{x}\log_b(1+t)^{1/t}
\end{aligned}$$

When $\delta x \to 0$, so $t \to 0$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \frac{1}{x} \lim_{t \to 0} \log_b(1+t)^{1/t} \\
&= \frac{1}{x} \log_b(\lim_{t \to 0}(1+t)^{1/t})
\end{aligned}$$

The following table shows $\lim_{t \to 0}(1+t)^{1/t} = ?$

t	$\frac{1}{t}$	$(1+t)^{1/t}$
1	1	2
0.5	2	2.25
0.25	4	2.4414
0.10	10	2.5937
0.01	100	2.7048
0.001	1000	2.7169
0.0001	10000	2.7181
0.00001	100000	2.7183
0.000001	1000000	2.7183

The above table shows that $(1+t)^{1/t} \to 2.7183$ as $t \to 0$.
This limit value is denoted by e which is called the **exponential number.**
In fact, e is an irrational number.
Therefore,

$$\frac{dy}{dx} = \frac{1}{x}\log_b e$$

$$\boxed{\frac{d}{dx}\log_b x = \frac{1}{x}\log_b e}$$

If $u(x) > 0$ is a function of x,

$$\boxed{\frac{d}{dx}\log_b u(x) = \frac{1}{u(x)} \cdot \log_b e \cdot \frac{d}{dx}u(x)}$$

Logarithm of x to the base e is called **natural** or Napierian Logarithm and denote $\log_e x = \ln x$.
Since

$$\frac{d}{dx}\log_b x = \frac{1}{x}\log_b e$$
$$\frac{d}{dx}\log_e x = \frac{1}{x}\log_e e$$

$$\therefore \boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

general $\boxed{\dfrac{d}{dx}\ln u(x) = \dfrac{1}{u(x)} \cdot \dfrac{du}{dx}}$.

xample 3.

ifferentiate the following functions with respect to x.

$\log_{10} x^3$ (b) $\log_2 x^3$ (c) $\ln x^3$ (d) $\ln\sqrt{x^2+5}$

$\ln \sin 2x$ (f) $\ln x \cdot \log_{10} x$ (g) $\ln\dfrac{x}{\sqrt{x^2+2}}$ (h) $\dfrac{x^2}{\log_{10} x}$

olution:

$$\frac{d}{dx}\log_{10} x^3 = \frac{1}{x^3}\cdot\log_{10} e\cdot\frac{d}{dx}x^3$$
$$= \frac{1}{x^3}\cdot\log_{10} e\cdot 3x^2$$
$$= \frac{3}{x}\log_{10} e$$

b)
$$\frac{d}{dx}\log_2 x^3 = \frac{1}{x^3}\cdot\log_2 e\cdot\frac{d}{dx}x^3$$
$$= \frac{1}{x^3}\cdot\log_2 e\cdot 3x^2$$
$$= \frac{3}{x}\log_2 e$$

c)
$$\frac{d}{dx}\ln x^3 = \frac{1}{x^3}\cdot\frac{d}{dx}x^3$$
$$= \frac{1}{x^3}\cdot 3x^2$$
$$= \frac{3}{x}$$

d)
$$\frac{d}{dx}\ln\sqrt{x^2+5} = \frac{1}{\sqrt{x^2+5}}\cdot\frac{d}{dx}\sqrt{x^2+5}$$
$$= \frac{1}{\sqrt{x^2+5}}\cdot\frac{1}{2\sqrt{x^2+5}}\cdot 2x$$
$$= \frac{x}{x^2+5}$$

e)
$$\frac{d}{dx}\ln\sin 2x = \frac{1}{\sin 2x}\cdot\frac{d}{dx}\sin 2x$$
$$= \frac{1}{\sin 2x}\cdot 2\cos 2x$$
$$= 2\cot 2x$$

(f)
$$\begin{aligned}\frac{d}{dx}\ln x\cdot\log_{10}x &= \ln x\cdot\frac{d}{dx}\log_{10}x+\log_{10}x\cdot\frac{d}{dx}\ln x\\ &= \ln x\cdot\frac{1}{x}\cdot\log_{10}e+\log_{10}x\cdot\frac{1}{x}\end{aligned}$$

(g)
$$\begin{aligned}\frac{d}{dx}\ln\frac{x}{\sqrt{x^2+2}} &= \frac{\sqrt{x^2+2}}{x}\cdot\frac{d}{dx}\frac{x}{\sqrt{x^2+2}}\\ &= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\sqrt{x^2+2}\frac{dx}{dx}-x\frac{d}{dx}\sqrt{x^2+2}}{x^2+2}\\ &= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\sqrt{x^2+2}-x\cdot\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+2}}}{x^2+2}\\ &= \frac{\sqrt{x^2+2}}{x}\cdot\frac{x^2+2-x^2}{(x^2+2)\cdot\sqrt{x^2+2}}\\ &= \frac{2}{x(x^2+2)}\end{aligned}$$

(h)
$$\begin{aligned}\frac{d}{dx}\frac{x^2}{\log_{10}x} &= \frac{\log_{10}x\cdot\frac{d}{dx}x^2-x^2\cdot\frac{d}{dx}\log_{10}x}{(\log_{10}x)^2}\\ &= \frac{\log_{10}x\cdot 2x-x^2\cdot\frac{1}{x}\cdot\log_{10}e}{(\log_{10}x)^2}\\ &= \frac{2x\log_{10}x-x\log_{10}e}{(\log_{10}x)^2}\end{aligned}$$

Logarithmic Differentiation: The derivatives of positive functions can be found by taking the natural logarithm of both sides before differentiating as in the following examples.

Example 4.

Find $\frac{dy}{dx}$ if $y=\sqrt{(x^2+1)(x-1)^2}$.

Solution

$$\begin{aligned}y &= \sqrt{(x^2+1)(x-1)^2}\\ \ln y &= \ln\sqrt{(x^2+1)(x-1)^2}\\ &= \frac{1}{2}(\ln(x^2+1)+\ln(x-1)^2)\\ \frac{d}{dx}\ln y &= \frac{d}{dx}(\frac{1}{2}(\ln(x^2+1)+\ln(x-1)^2))\end{aligned}$$

$$
\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}(\frac{1}{x^2+1}\cdot 2x + \frac{1}{(x-1)^2}\cdot 2(x-1)) \\
&= \frac{x}{x^2+1} + \frac{2}{x-1} \\
\frac{dy}{dx} &= y(\frac{x}{x^2+1} + \frac{2}{x-1}) \\
&= \sqrt{(x^2+1)(x-1)^2}\,(\frac{x}{x^2+1} + \frac{2}{x-1})
\end{aligned}
$$

Example 5.

Find $\frac{dy}{dx}$ if $y = \frac{x\sin x}{\sqrt{\sec x}}, \quad 0 < x < \frac{\pi}{2}$.

Solution

$$
\begin{aligned}
y &= \frac{x\sin x}{\sqrt{\sec x}}, \quad 0 < x < \frac{\pi}{2} \\
\ln y &= \ln(\frac{x\sin x}{\sqrt{\sec x}}) \\
&= \ln x + \ln\sin x - \frac{1}{2}\ln\sec x \\
\frac{d}{dx}\ln y &= \frac{d}{dx}(\ln x + \ln\sin x - \frac{1}{2}\ln\sec x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{\sin x}\cdot\cos x - \frac{1}{2}\frac{1}{\sec x}\cdot\sec x\cdot\tan x \\
&= \frac{1}{x} + \cot x - \frac{1}{2}\tan x \\
\frac{dy}{dx} &= y(\frac{1}{x} + \cot x - \frac{1}{2}\tan x) \\
&= \frac{x\sin x}{\sqrt{\sec x}}(\frac{1}{x} + \cot x - \frac{1}{2}\tan x)
\end{aligned}
$$

Exercise 8.2

1. Differentiate the following functions with respect to x.

(a) $\ln(2x^2+3)$ (b) $\ln|x|$ (c) $x^2\log_2 x$

(d) $\sin 3x\cdot\log_{10}(x+1)$ (e) $\ln\sqrt{5x-4}$ (f) $\ln(\ln x)$

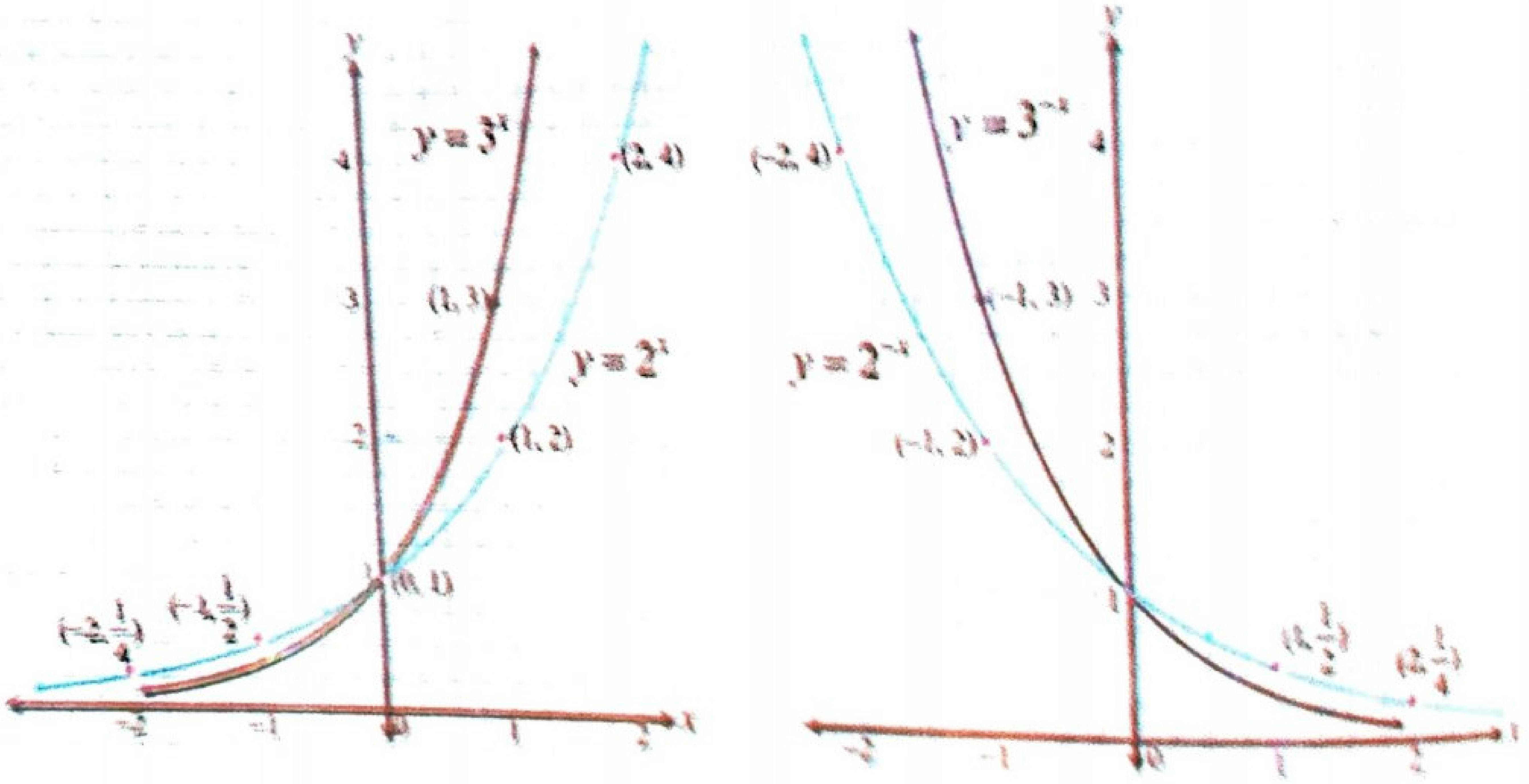
2. Use logarithmic differentiation to find the derivative of y with respect to x.

(a) $y = \sqrt{x(x+1)}$ (b) $y = x(x+1)(x+2)$ (c) $y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$

8.3 Exponential Functions

A function of the form $y = b^x$, where $b > 0$, $b \neq 1$, is called an **exponential function** of x.

The graphs $y = 2^x$, $y = 3^x$, $y = 2^{-x}$, and $y = 3^{-x}$ are as follows.



Domain of $y = b^x$: R Range: $x > 0$ Asymptote: x-axis ($y = 0$)

Note that $y = \log_b x$ and $y = b^x$ are inverse of each other.

For $b > 1$, $y = b^x$ is an **exponential growth function** as the functions $y = 2^x$ and $y = 3^x$.

For $0 < b < 1$, $y = b^x$ is an **exponential decay function** as the functions $y = 2^{-x} = (\frac{1}{2})^x$ and $y = 3^{-x} = (\frac{1}{3})^x$.

$$\boxed{y = b^x \xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = b^{-x}}$$

From the graph of $y = b^x$, the graph of $y = ab^x$, $a > 0$ can be obtained as

$$\boxed{y = b^x \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} y = ab^x}$$

From the graph of $y = ab^x$, $a > 0$, the graph of $y = ab^{x-h} + k$ can be obtained as

$$\boxed{y = ab^x \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} y = ab^{x-h} + k}$$

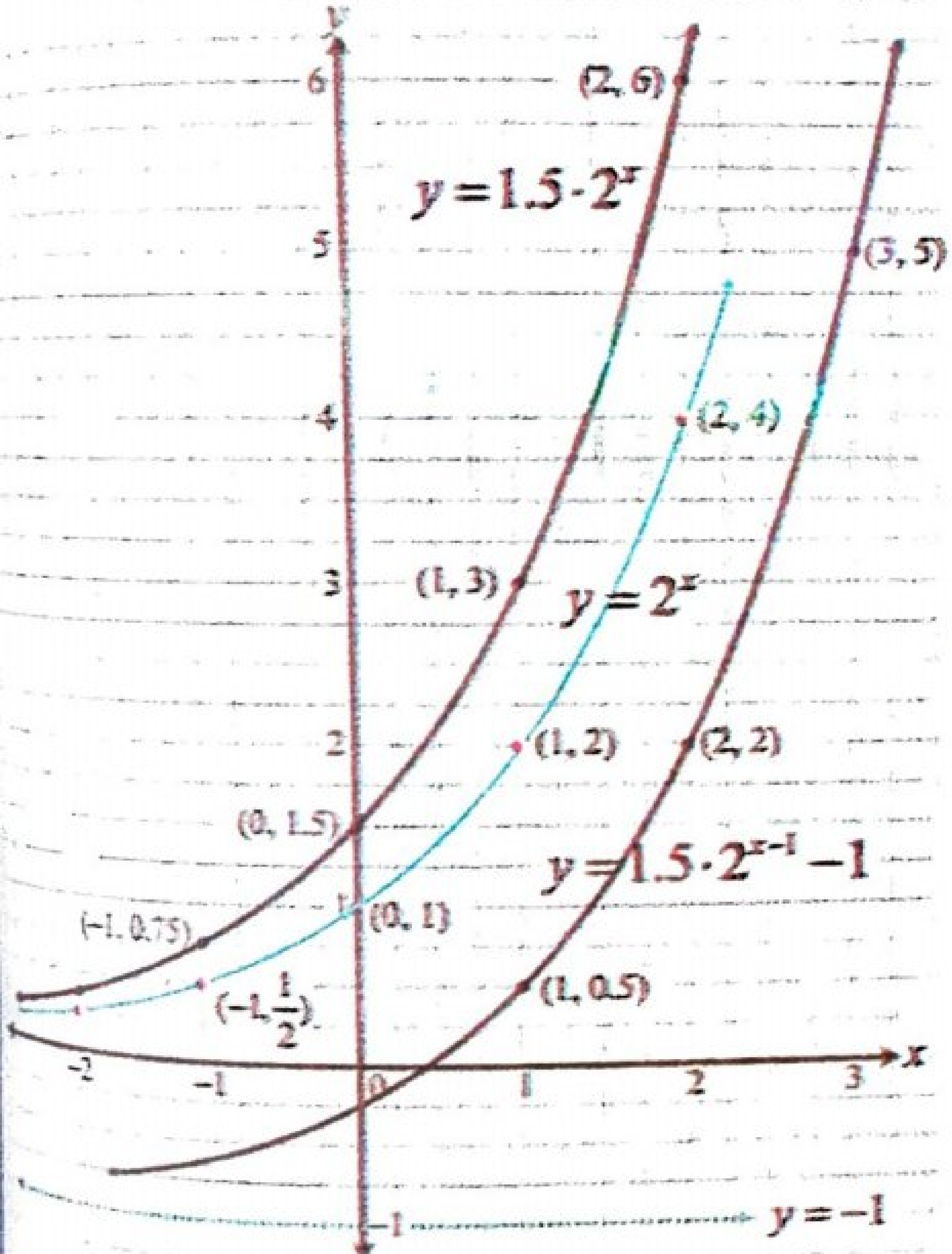
ple 6.

the graphs of $y = 1.5\cdot 2^{x-1} - 1$ from $y = 2^x$, and $y = -2^{-(x+1)} + 2$ from $y = -2^{-x}$.

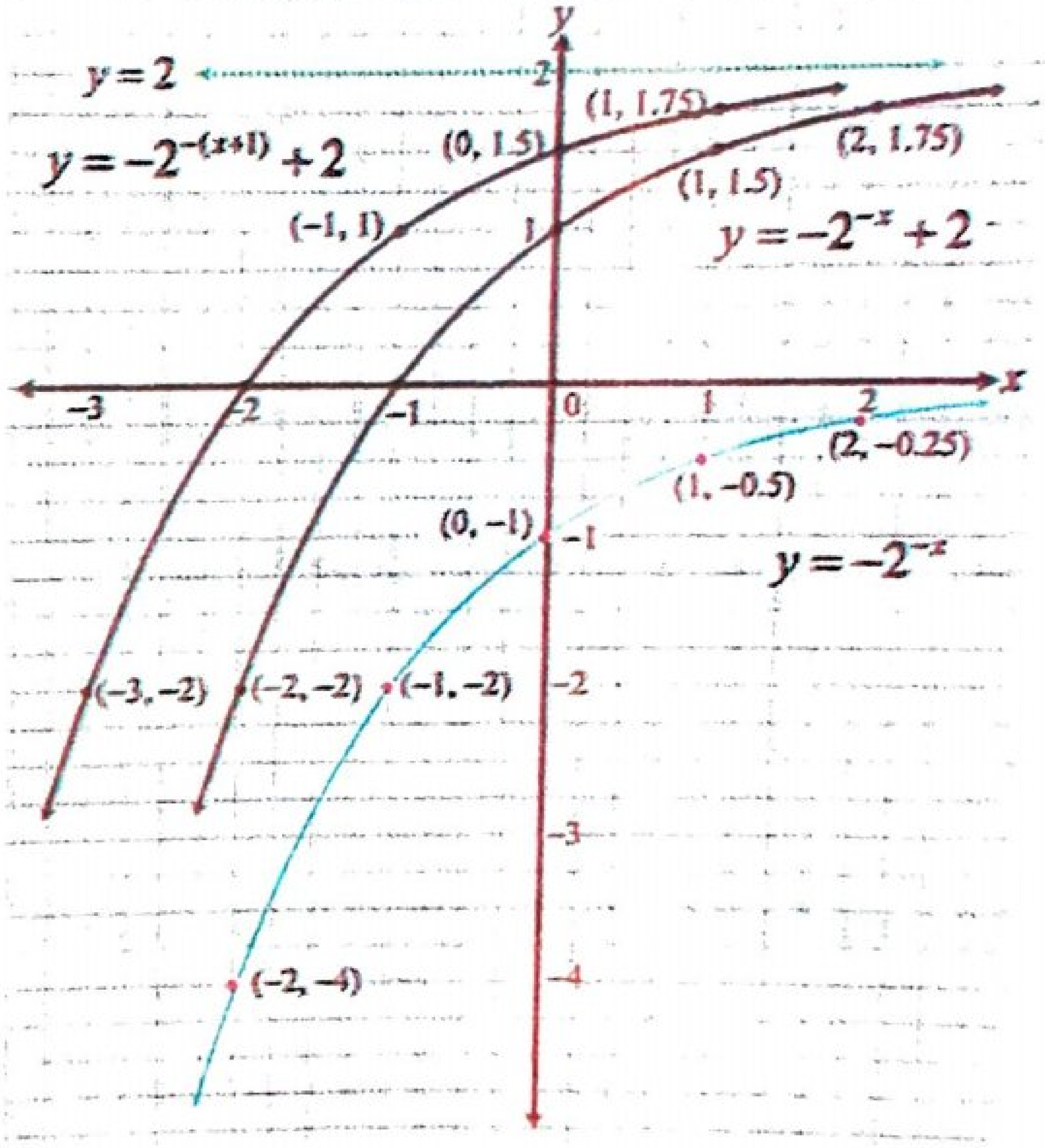
ion

$$y = 2^x \xrightarrow[\text{scale factor } 1.5]{\text{vertical scaling}} y = 1.5\cdot 2^x \xrightarrow[\text{vertical translation } -1 \text{ unit}]{\text{horizontal translation 1 unit}} y = 1.5\cdot 2^{x-1} - 1$$

$$y = -2^{-x} \xrightarrow[\text{2 units}]{\text{vertical translation}} y = -2^{-x} + 2 \xrightarrow[-1 \text{ unit}]{\text{horizontal translation}} y = -2^{-(x+1)} + 2$$



Note: Asymptote $y = -1$



Note: Asymptote $y = 2$

ample 7..

ts $(0,1)$ and $(1,b)$ are on the graph of $y = b^x$. Find the corresponding points on graphs of $y = ab^x$ and $y = ab^{x-h} + k$. What is the asymptote of $y = ab^{x-h} + k$? d the range of $y = ab^{x-h} + k$ if $a > 0$ and if $a < 0$.

ution

$$y = b^x \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} y = ab^x \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} y = ab^{x-h} + k$$

$$(0,1) \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} (0,a) \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} (h, a+k)$$

$$(1,b) \xrightarrow[\text{scale factor } a]{\text{vertical scaling}} (1,ab) \xrightarrow[\text{vertical translation } k \text{ units}]{\text{horizontal translation } h \text{ units}} (1+h, ab+k)$$

mptote of $y = b^x$ is $y = 0$ (x-axis) and asymptote of $y = ab^x$ is also $y = 0$ (x-axis). er vertical translation k units, asymptote of $y = ab^{x-h} + k$ is $y = k$. If $a > 0$, then the range of $y = ab^{x-h} + k$ is $\{y : y > k\}$ that means all of y above the $y = k$. If $a < 0$, then the range of $y = ab^{x-h} + k$ is $\{y : y < k\}$ that means below the asymptote $y = k$.

Exercise 8.3

1. Draw the graph of (a) $y = 2 \cdot 3^{x+1} - 2$ (b) $y = -2^{x-1} + 3$.
2. Draw the graph of (a) $y = 2^{|x|}$ (b) $y = 2^{-|x|}$
3. Find the y-intercept, asymptote and the range of
   (a) $y = 3e^{x-1} + 2$ (b) $y = -2e^{-x+1} + 3$.

8.4 Differentiation of Exponential Functions

Derivative of $y = b^x$, $b > 0$, $b \neq 1$

$$y = b^x, \quad b > 0,\ b \neq 1$$
$$\therefore x = \log_b y$$

Differentiate both sides with respect to x,

$$1 = \frac{1}{y} \log_b e \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = y \cdot \frac{1}{\log_b e} = b^x \log_e b = b^x \ln b$$

Therefore $\boxed{\frac{d}{dx} b^x = b^x \ln b}$

Since $\ln e = 1$, $\boxed{\frac{d}{dx} e^x = e^x}$ and $\boxed{\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot \frac{d}{dx} u(x)}$

In general $\boxed{\frac{d}{dx} b^{u(x)} = b^{u(x)} \cdot \ln b \cdot \frac{d}{dx} u(x)}$

Example 8.

Differentiate the following functions with respect to x.

(a) e^{3x} (b) e^{1-x^2} (c) $e^{\sin x}$ (d) $x^2 e^{3x}$

(e) $e^{2x} \sin 3x$ (f) $(e^x + e^{-x})^2$ (g) $\frac{3e^{2x}}{1-2x}$

ion

$$\frac{d}{dx}e^{3x} = e^{3x} \cdot \frac{d}{dx}3x = 3e^{3x}$$

$$\frac{d}{dx}e^{1-x^2} = e^{1-x^2} \cdot \frac{d}{dx}(1-x^2) = e^{1-x^2}(-2x)$$

$$\frac{d}{dx}e^{\sin x} = e^{\sin x} \cdot \frac{d}{dx}\sin x = e^{\sin x}\cos x$$

$$\begin{aligned}\frac{d}{dx}x^2e^{3x} &= x^2 \cdot \frac{d}{dx}e^{3x} + e^{3x} \cdot \frac{d}{dx}x^2\\ &= x^2 \cdot e^{3x} \cdot 3 + e^{3x} \cdot 2x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}e^{2x}\sin 3x &= e^{2x} \cdot \frac{d}{dx}\sin 3x + \sin 3x \cdot \frac{d}{dx}e^{2x}\\ &= e^{2x} \cdot \cos 3x \cdot 3 + \sin 3x \cdot e^{2x} \cdot 2\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(e^x + e^{-x})^2 &= 2(e^x + e^{-x}) \cdot \frac{d}{dx}(e^x + e^{-x})\\ &= 2(e^x + e^{-x})(e^x - e^{-x})\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\frac{3e^{2x}}{1-2x} &= 3\,\frac{(1-2x) \cdot \frac{d}{dx}e^{2x} - e^{2x} \cdot \frac{d}{dx}(1-2x)}{(1-2x)^2}\\ &= 3\,\frac{(1-2x) \cdot 2e^{2x} - e^{2x}(-2)}{(1-2x)^2}\\ &= 3\,\frac{(4-4x)e^{2x}}{(1-2x)^2}\end{aligned}$$

mple 9.

$\frac{dy}{dx}$.

$= e^x \ln x$ (b) $y = \log_{10} e^{x^2}$

$= \log_3(\sin x + e^x)$ (d) $xe^y + \ln(xy) = \sin x$

tion

$$\begin{aligned}y &= e^x \ln x\\ \frac{dy}{dx} &= e^x \frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}e^x\\ &= e^x \cdot \frac{1}{x} + \ln x \cdot e^x\end{aligned}$$

(b)
$$y=\log_{10} e^{x^2}=x^2\log_{10} e$$
$$\frac{dy}{dx}=\log_{10} e\cdot\frac{d}{dx}x^2$$
$$=\log_{10} e\cdot 2x$$

(c)
$$y=\log_5(\sin x+e^x)$$
$$\frac{dy}{dx}=\frac{1}{\sin x+e^x}\cdot\log_5 e\cdot\frac{d}{dx}(\sin x+e^x)$$
$$=\frac{1}{\sin x+e^x}\cdot\log_5 e(\cos x+e^x)$$

(d)
$$xe^y+\ln(xy)=\sin x$$

Differentiate both sides with respect to x.

$$x\cdot e^y\cdot\frac{dy}{dx}+e^y+\frac{1}{xy}\left(x\frac{dy}{dx}+y\right)=\cos x$$
$$\left(xe^y+\frac{1}{y}\right)\frac{dy}{dx}=\cos x-e^y-\frac{1}{x}$$
$$\frac{dy}{dx}=\frac{\cos x-e^y-\frac{1}{x}}{xe^y+\frac{1}{y}}$$

Example 10.

Differentiate $y=x^x$, $x>0$.

Solution

$$y=x^x$$
$$y=e^{\ln x^x}$$
$$y=e^{x\ln x}$$
$$\frac{dy}{dx}=\frac{d}{dx}e^{x\ln x}$$
$$=e^{x\ln x}\frac{d}{dx}(x\ln x)$$
$$=e^{x\ln x}\left(x\cdot\frac{1}{x}+\ln x\cdot 1\right)$$
$$=x^x(1+\ln x)$$

Exercise 8.4

1. Differentiate the following functions with respect to x.

(a) $(3+3x)e^{-3x}$
(b) $3^x x^3$
(c) $2^x \log_2 x$
(d) $10^x \log_{10}(x+1)$
(e) $\frac{x^2+\tan 3x}{e^x}$
(f) $x\ln y + e^{xy} = 2$

2. Given that $y = e^{3x}\sin 2x$, prove that $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

3. Use logarithmic differentiation to find the derivative of y with respect to x.

(a) $y = (\sqrt{x})^x$
(b) $y = x^{\cos x}$
(c) $y = (\ln x)^{\ln x}$

Chapter 9

Application of Differentiation

We have studied the derivatives of functions such as polynomial, rational, exponential, logarithmic and trigonometric functions. We now are ready to apply these derivatives in:

- finding the equations of tangent and normal lines,
- finding the critical points to analyze the maximum, minimum and inflection points,
- testing the concavity,
- finding the maxima and minima,
- solving the approximation problems.

9.1 Tangent Line and Normal Line

We first introduce the inclination of the line and its slope. The **inclination** of a line that intersects the x-axis is the measure of the smallest nonnegative angle which the line makes with the positive end of the x-axis. We shall use the symbol θ to represent the angle of inclination.

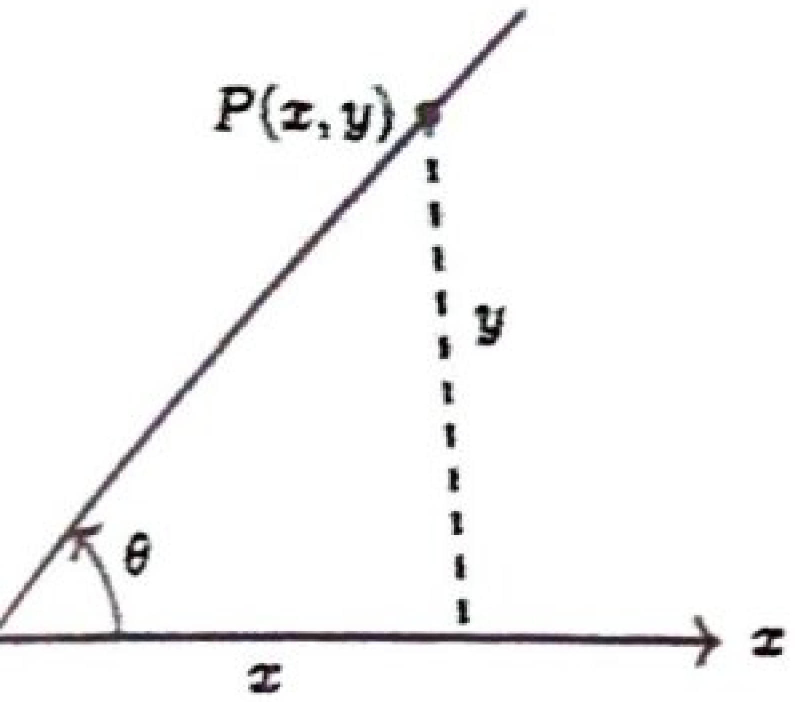
We here discuss the expression related to the inclination, namely, the slope of a line, says m, is the tangent of the inclination,

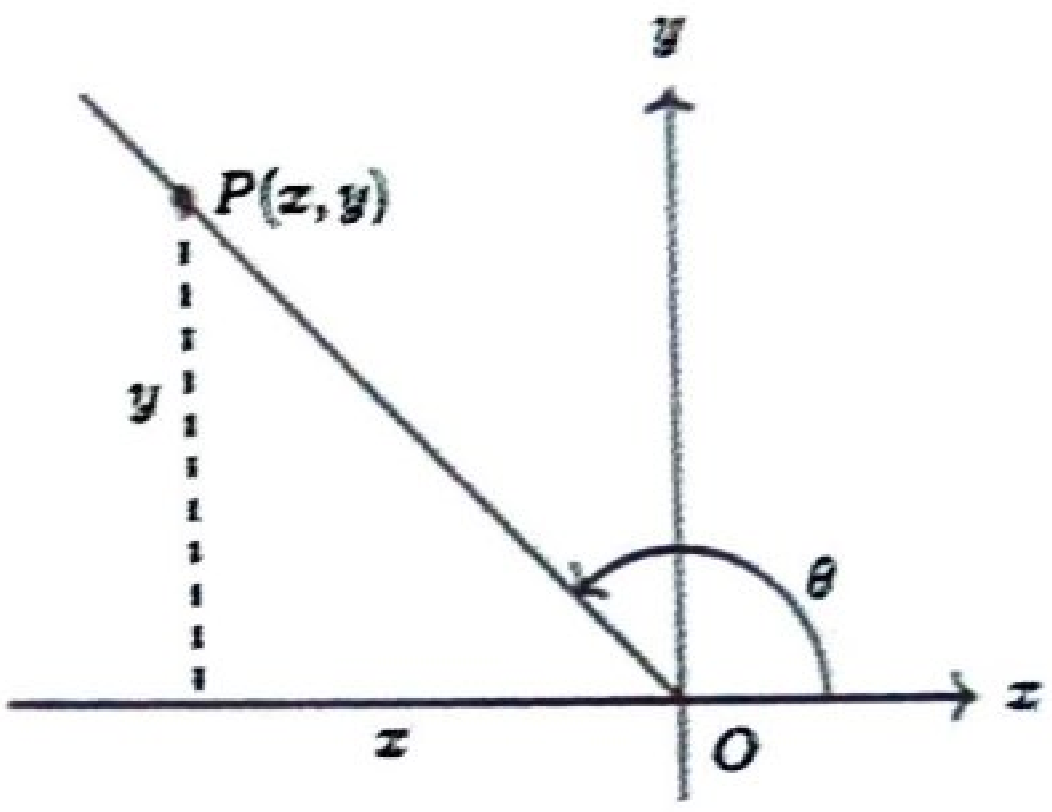
$$m = \tan\theta.$$

We see that the vertical lines have inclination 90° but no slope. A horizontal line has a slope and that slope is the number 0.

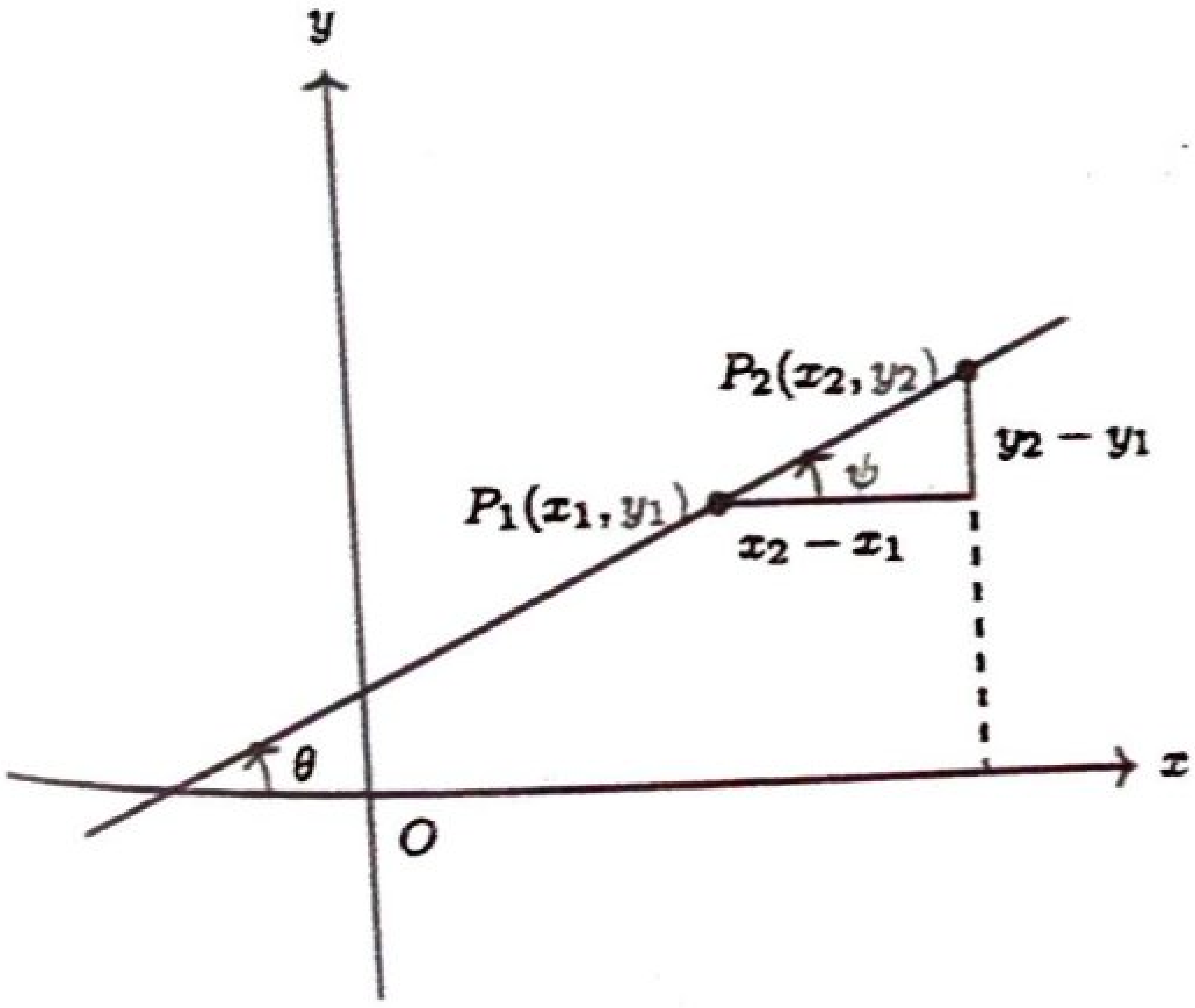
as shown in either of the two positions in the following figures, then

$$\tan\theta = \frac{y}{x}.$$


$P(x,y)$
y
θ
x
x


y
$P(x,y)$
y
θ
x
O
x

ppose we have a line with a pair of points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, on it.

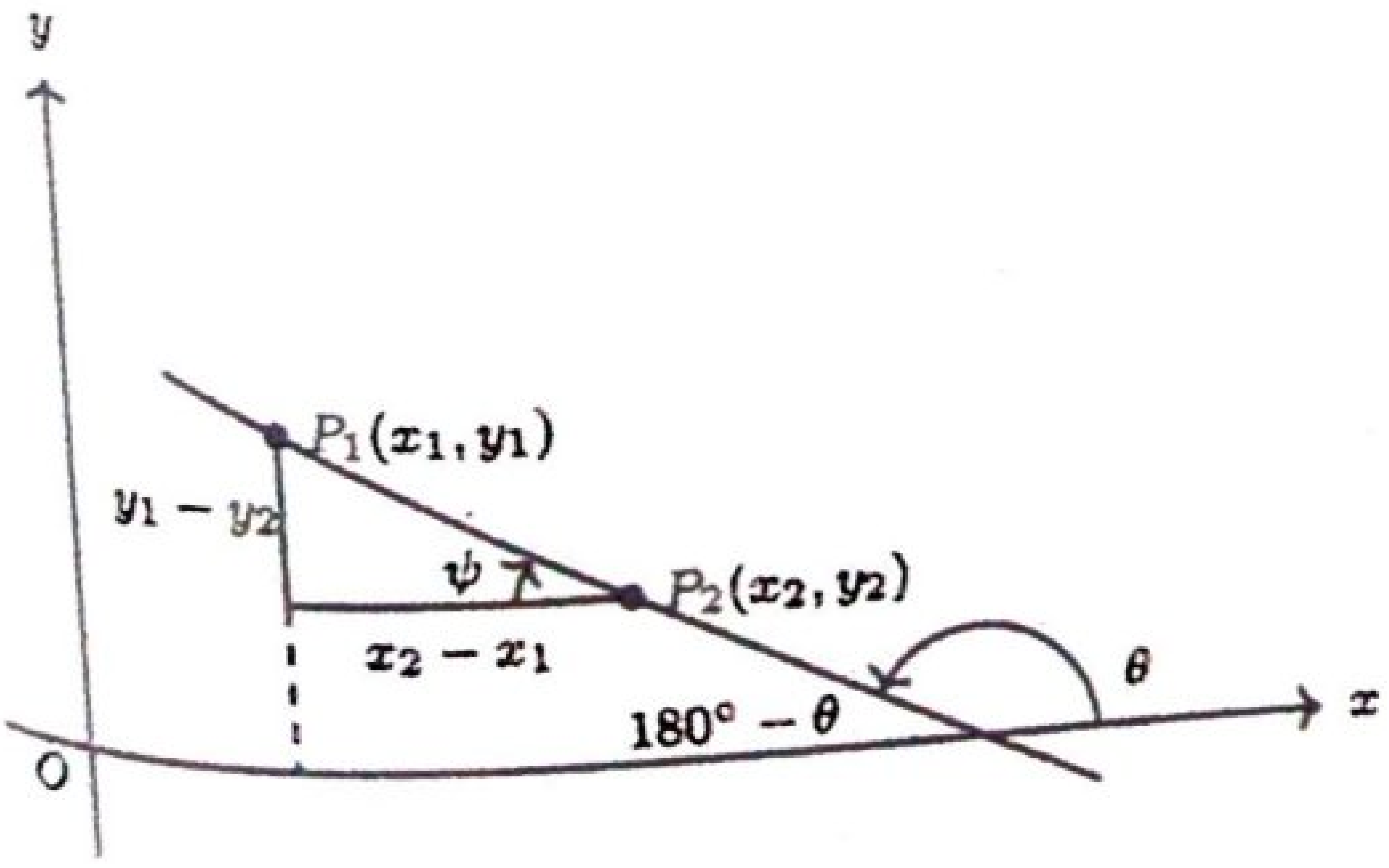

y
$P_2(x_2,y_2)$
$y_2 - y_1$
ψ
$P_1(x_1,y_1)$
$x_2 - x_1$
θ
O
x

From the figure, we can write as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\psi.$$

Since $\theta = \psi$, we get

$$m = \tan\theta.$$


y
$P_1(x_1,y_1)$
$y_1 - y_2$
ψ
$P_2(x_2,y_2)$
$x_2 - x_1$
θ
$180° - \theta$
O
x

From the figure, we can write as

$$m = \frac{y_1 - y_2}{x_2 - x_1} = -\frac{y_2 - y_1}{x_2 - x_1}$$
$$= -\tan\psi.$$

Since $180° - \theta = \psi$, we get

$$m = -\tan\psi = -\tan(180° - \theta)$$
$$= \tan\theta.$$

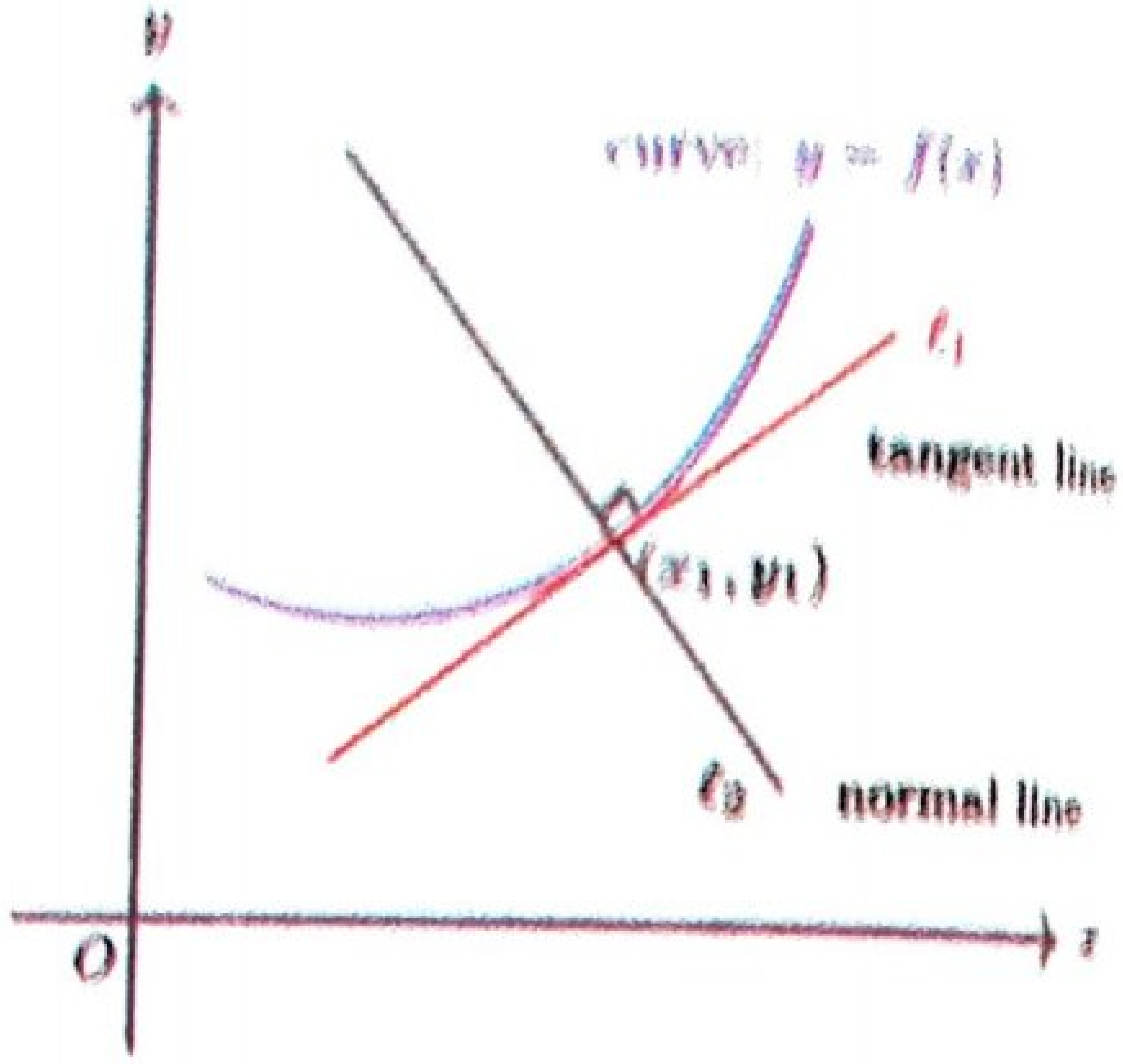
Next we recall the notion of equations of tangent and normal line.

For the curve $y = f(x)$, the gradient of the tangent ℓ_1 at the point (x_1, y_1) is the value of $y'(x)$ at $x = x_1$, hence the **equation of the tangent** at (x_1, y_1) is

$$y - y_1 = y'(x_1)(x - x_1).$$

The line ℓ_2 which is perpendicular to the tangent ℓ_1 at (x_1, y_1) is called the **normal** to the curve at (x_1, y_1). Hence its gradient is the value of $-(y'(x_1))^{-1}$ where $y'(x_1) \neq 0$ and the **equation of the normal** at (x_1, y_1) is

$$y - y_1 = -\frac{1}{y'(x_1)}(x - x_1).$$



Example 1.

Find the equations of tangent and normal line to the curve $y = \ln x$ at the point $y = -1$.

Solution

$$y = \ln x,$$

$$y' = \frac{1}{x}.$$

When $y = -1$, $\ln x = -1$,

$$x = e^{-1} = \frac{1}{e}.$$

At the point $(\frac{1}{e}, -1)$, the gradient of tangent is $y'(\frac{1}{e}) = e$, and

the gradient of normal is $-\frac{1}{e}$.

The equation of tangent at $(\frac{1}{e}, -1)$ is

$$y - (-1) = e(x - \frac{1}{e})$$

$$y = ex - 2.$$

The equation of normal at $(\frac{1}{e}, -1)$ is

$$y - (-1) = -\frac{1}{e}(x - \frac{1}{e})$$

$$y = -\frac{1}{e}x + \frac{1}{e^2} - 1.$$

Example 2.

Find the equation of tangent to $y = \sin x$ at the origin.

Solution

$$y = \sin x,$$
$$y' = \cos x.$$

At $(0,0)$, the gradient of tangent is $y'(0) = \cos 0 = 1$.
The equation of tangent is

$$y - 0 = 1(x - 0)$$
$$y = x.$$

Example 3.

Consider the curve $f(x) = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point $x = 2$ is $y = 2x + 1$. Find the values of a and b.

Solution

$$f(x) = x^2 + ax + b,$$
$$f'(x) = 2x + a.$$

At $x = 2$, $f(2) = 2a + b + 4$ and $f'(2) = a + 4$.

By problem, the gradient of tangent line is 2, so $a + 4 = 2$ which gives $a = -2$.

The tangent line equation at $(2, 2a + b + 4)$ is

$$y - (2a + b + 4) = (a + 4)(x - 2).$$

Substitute the value $a = -2$ in above equation, then

$$y - b = 2(x - 2)$$
$$y = 2x + b - 4.$$

Since the tangent line equation at $x = 2$ is $y = 2x + 1$, it follows that

$$b = 5.$$

Therefore $a = -2$ and $b = 5$.

Exercise 9.1

1. Find the equations of tangent and normal line to the curve $y = 2\ln x$ at $(1,0)$.
2. Find the equation of a line which passes through the origin and tangent to $y = e^x$.

3. Find the points of contact where horizontal tangent meet the curve $y = x^4 - 2x^2 + 2$.

4. Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point at $x = 4$ is $4x + y = 22$. Find the values of a and b.

9.2 Maximum and Minimum

Before we describe the critical points let us discuss the behavior of increasing and decreasing functions.

Increasing and Decreasing Functions

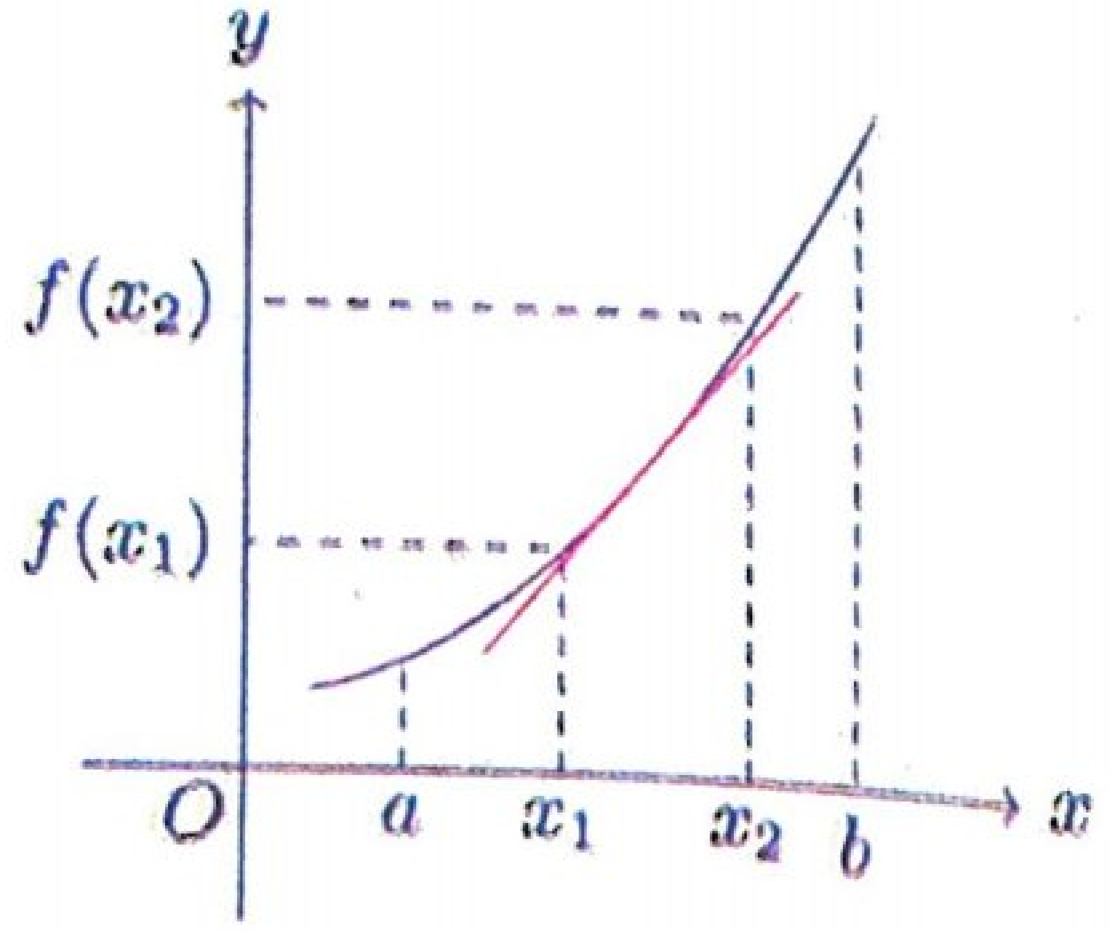
Consider the function on an open interval.

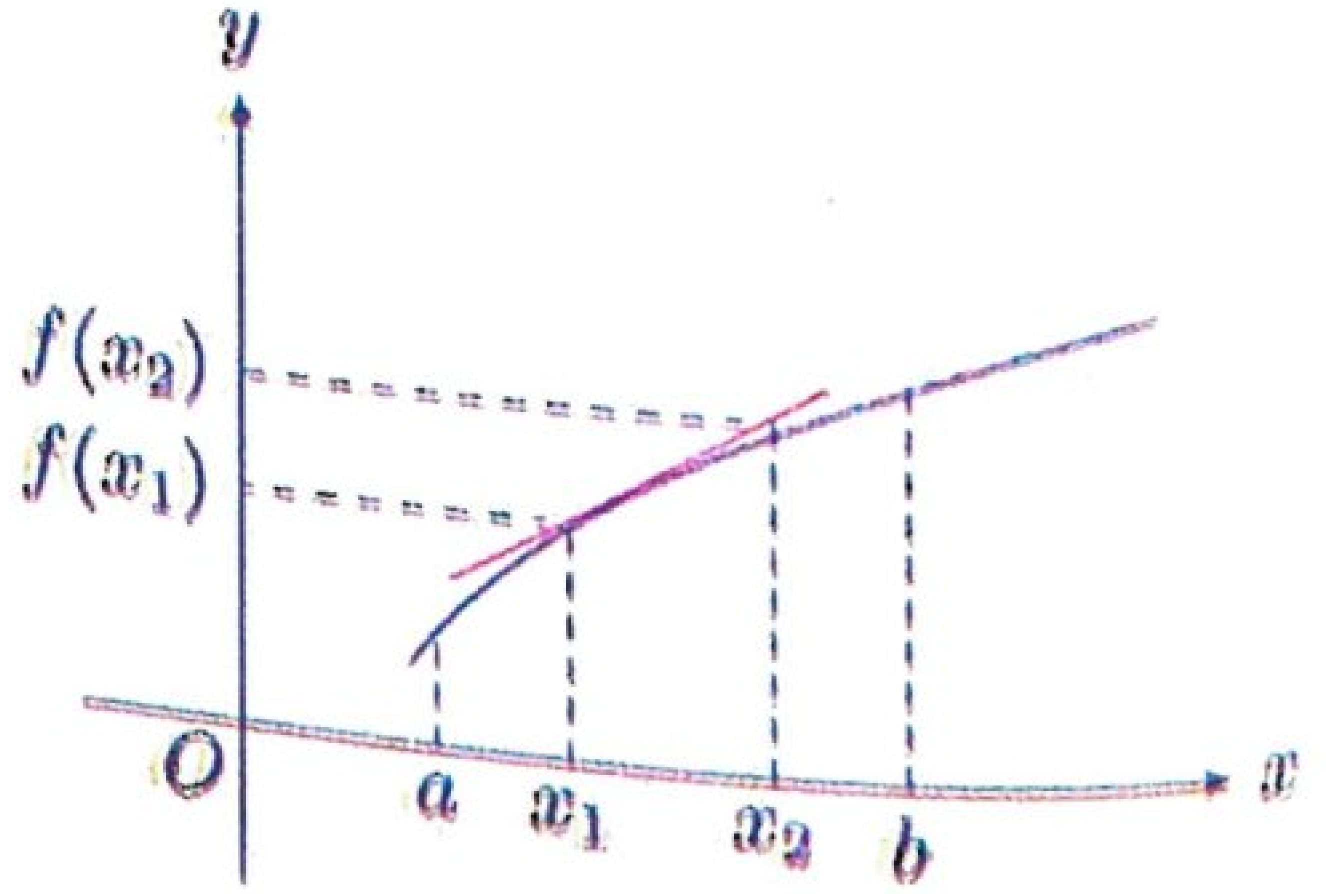
Definition.

A function f is **increasing** on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) > f(x_1),$$

namely, an increase in x produces an increase in y.

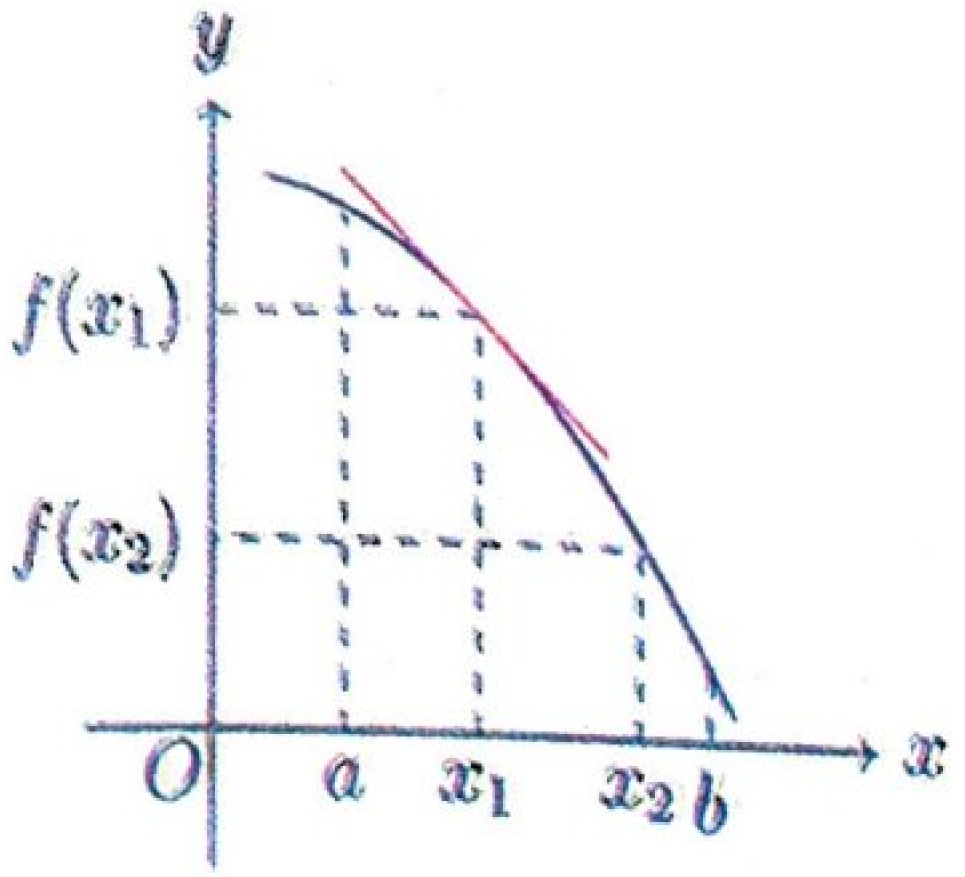


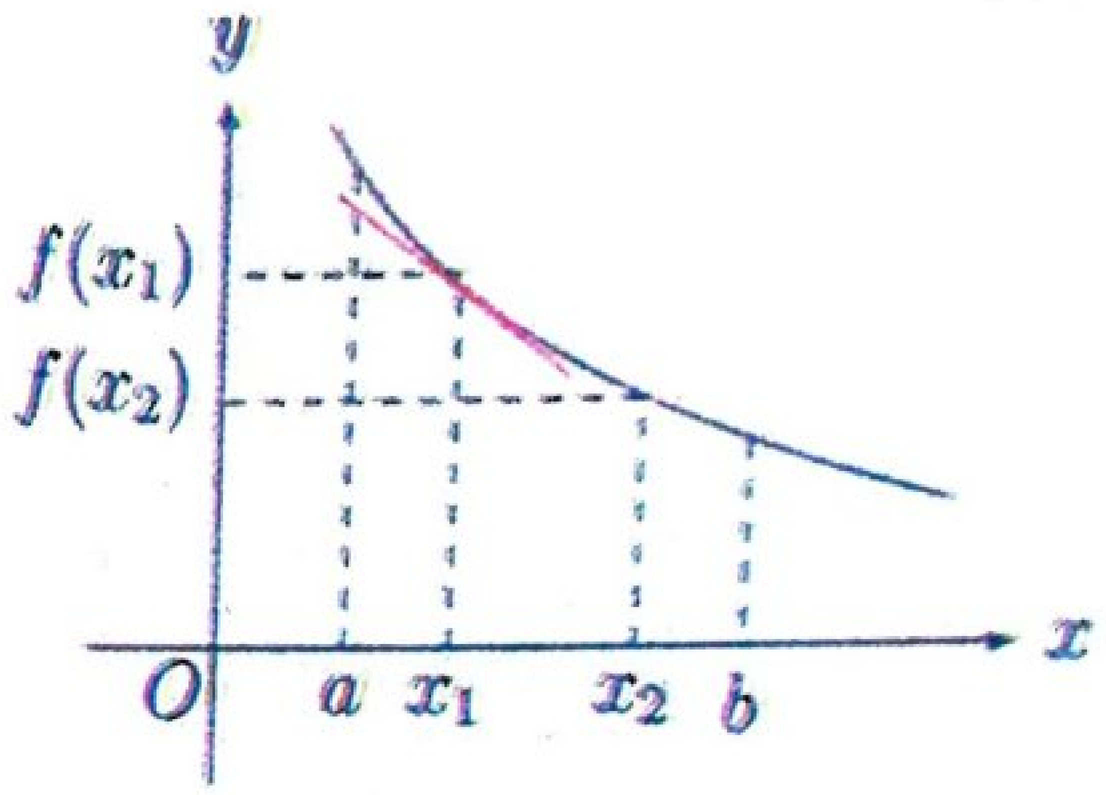


A function f is **decreasing** on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) < f(x_1),$$

namely, an increase in x produces a decrease in y.





We can determine whether a function is increasing or decreasing at a given interval by looking into its graph. Sometimes we can get the same information by using the first derivative.

Test for Increasing and Decreasing Functions

Let $y = f(x)$ be differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for each $x \in (a, b)$, then f is increasing on (a, b).
2. If $f'(x) < 0$ for each $x \in (a, b)$, then f is decreasing on (a, b).

Example 4.

Show that the function $f(x) = x^2 + 1$ is decreasing on the interval $x < 0$ and increasing on the interval $x > 0$.

Solution

$$f(x) = x^2 + 1,$$
$$f'(x) = 2x.$$

On $x < 0$, $f'(x) = 2x < 0$. Thus f is decreasing on $x < 0$.

On $x > 0$, $f'(x) = 2x > 0$. Thus f is increasing on $x > 0$.

Example 5.

Consider the function $f(x) = x^3 - 3x + 1$.

(a) Find the open intervals on which f is increasing and those on which f is decreasing.

(b) Find the points at which the graph has a horizontal tangent.

Solution

(a)

$$f(x) = x^3 - 3x + 1,$$
$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

So $f'(x) > 0$, when $x - 1$ and $x + 1$ have the same sign. Similarly, $f'(x) < 0$, when

$x-1$ and $x+1$ have the opposite signs.

Interval	$x<-1$	$-1<x<1$	$x>1$
$x+1$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
sign of $f'(x)$	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

Thus $f(x)$ is increasing on $x<-1$ and $x>1$, and decreasing on $-1<x<1$.

(b) The graph has a horizontal tangent when $f'(x)=0$.

$$3(x-1)(x+1)=0.$$

$$x=1 \text{ or } x=-1.$$

So $f(-1)=3$, $f(1)=-1$.

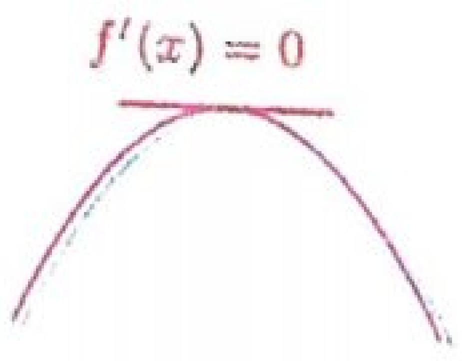
The horizontal tangents are at the points $(-1,3)$ and $(1,-1)$.

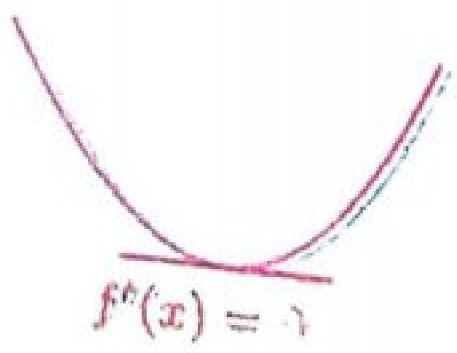
Critical Point

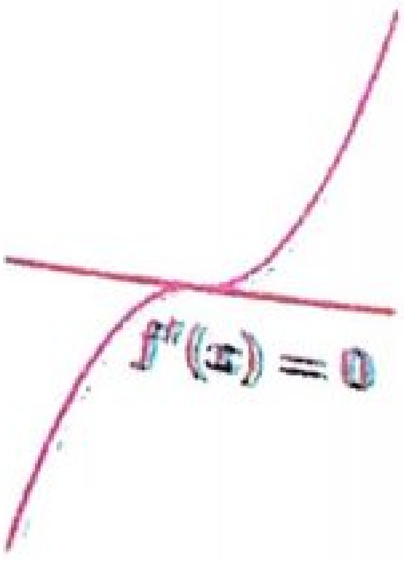
We now describe how to decide the maximum and minimum points of the function by determining the values of its derivatives. It is one of the most important applications of differentiation. We first introduce the notion of critical point.

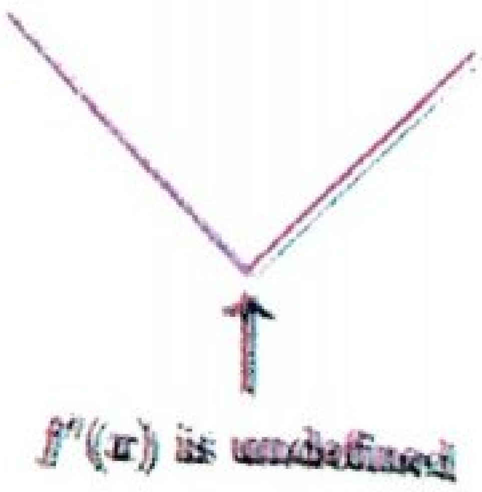
Definition.

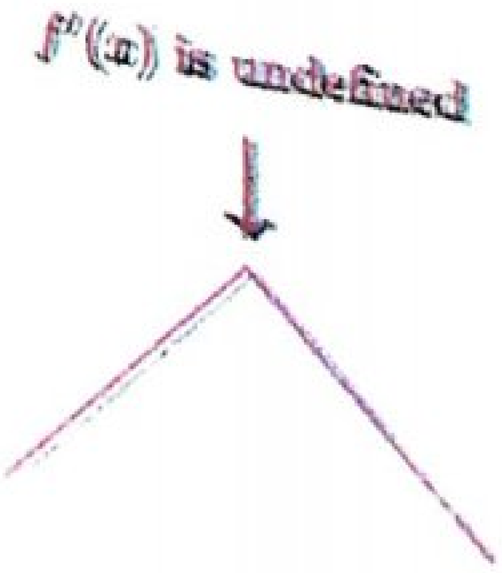
A **critical point** of a function $f(x)$ is a point in the domain of the function such that either $f'(x)=0$ or $f'(x)$ is undefined.











Test for Increasing and Decreasing Function on an Interval

Let $f(x)$ be a function.

Step 1 Find $f'(x)$.

Step 2 Find the critical points of f.

Step 3 Find the open intervals determining by the critical points.

Step 4 Test the signs of $f'(x)$ for each open interval.

Step 5 Decide whether f is increasing or decreasing on each interval.

Example 6.

Find the critical points of:

(a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = (x-1)^{\frac{2}{3}}$ (c) $f(x) = \frac{1}{x}$

Solution

(a)

curve:
$$f(x) = x^3 - 3x + 2,$$
$$f'(x) = 3x^2 - 3.$$

For critical points, $f'(x) = 0$ so $3x^2 - 3 = 0$.

Therefore $x = \pm 1$.

When $x = 1$, $f(1) = 1^3 - 3(1) + 2 = 0$.

When $x = -1$, $f(-1) = (-1)^3 - 3(-1) + 2 = 4$.

Therefore the critical points are $(1,0)$ and $(-1,4)$.

(b)

curve:
$$f(x) = (x-1)^{\frac{2}{3}},$$
$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}.$$

It is observed that $f'(x)$ is undefined when $x = 1$. Therefore the critical point is $(1,0)$.

(c)

curve:
$$f(x) = \frac{1}{x},$$
$$f'(x) = -\frac{1}{x^2}.$$

It is observed that $f'(x)$ is undefined when $x = 0$.

However, $x = 0$ is not in the domain of the function.
Therefore, there are no critical points.

Example 7.

Find the open intervals on which the following functions are increasing or decreasing.

(a) $f(x) = 2x^3 - 6x + 5$ (b) $f(x) = (x^2 - 9)^{\frac{4}{5}}$ (c) $f(x) = \dfrac{2x - 3}{x - 1}$

Solution

(a) curve:

$$f(x) = 2x^3 - 6x + 5,$$
$$f'(x) = 6(x^2 - 1) = 6(x + 1)(x - 1).$$

For critical points, $f'(x) = 0$, namely $x = \pm 1$.

Interval	$x < -1$	$-1 < x < 1$	$x > 1$
$x + 1$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
sign of $f'(x)$	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

Therefore, on $x < -1$ and $x > 1$, the given function is increasing, and on $-1 < x <$ the given function is decreasing.

(b) curve:

$$f(x) = (x^2 - 9)^{\frac{4}{5}},$$
$$f'(x) = \frac{8}{5}\,\frac{x}{(x^2 - 9)^{\frac{1}{5}}}.$$

For critical points, $f'(x) = 0$, namely $x = 0$.

$f'(x)$ is undefined, namely $x = -3$ and $x = 3$.

Interval	$x < -3$	$-3 < x < 0$	$0 < x < 3$	$x > 3$
$x + 3$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x - 3$	$-$	$-$	$-$	$+$
sign of $f'(x)$	$-$	$+$	$-$	$+$
Behavior of f	decreasing	increasing	decreasing	increasing

Therefore, on $x < -3$ and $0 < x < 3$, the given function is decreasing, and on $-3 < x < 0$ and $x > 3$ the given function is increasing.

curve:

$$f(x) = \frac{2x-3}{x-1},$$

$$f'(x) = \frac{1}{(x-1)^2}.$$

It is observed that $x = 1$ does not belong to the domain of function.

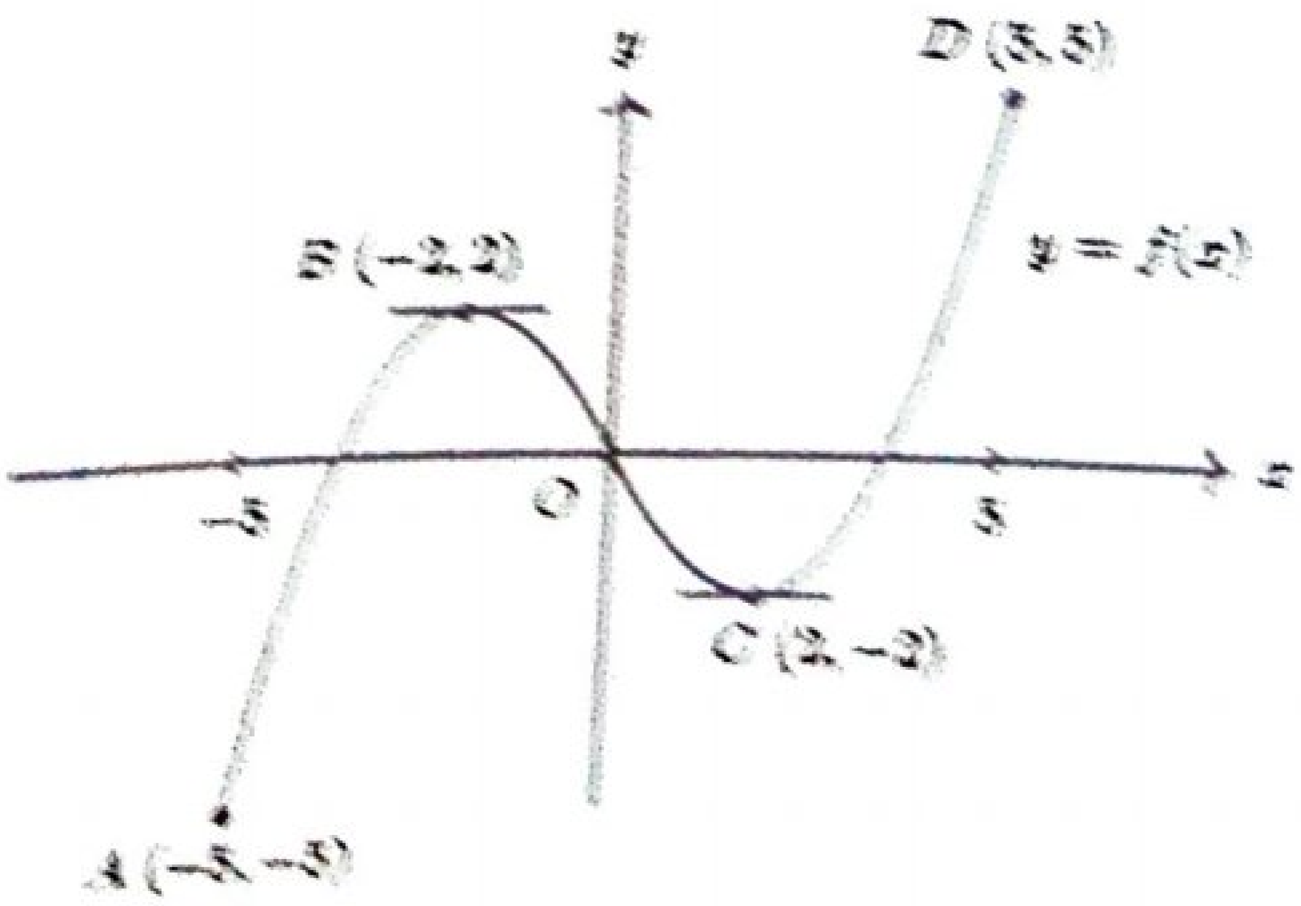
Interval	$x < 1$	$x > 1$
$(x-1)^2$	$+$	$+$
sign of $f'(x)$	$+$	$+$
Behavior of f	increasing	increasing

Therefore, on both $x < 1$ and $x > 1$, the function is increasing.

First Derivative Test

Consider the following graph which has a restricted domain of $-5 \leq x \leq 5$.

In the figure, points A and C are minimum, and points B and D are maximum. The point A is called a **global minimum** because $f(x)$ has a minimum value on the entire domain. Similarly the point D is called a **global maximum** because $f(x)$ has a maximum value on the entire domain. B is a **local maximum** because it is a turning point on the graph and $f(x)$ has a maximum value at B compared to the neighboring points. C is a **local minimum** because it is a turning point on the graph and $f(x)$ has a minimum value at C compared to the neighboring points.



Test for Global Maximum and Global Minimum on a Closed Interval

Let $f(x)$ be a continuous function on a closed interval.

Step 1 Find the critical points of f.

Step 2 Find $f(x)$ at all critical points and endpoints of the interval.

Step 3 Take the smallest value as the global minimum and the largest value as the global maximum.

Example 8.

Find the global minimum and global maximum values of each of the following function:

(a) $f(x) = x^2 + 5x - 3$ on $[-4, 5]$, (b) $f(x) = \dfrac{x}{x^2+1}$ on $[-3, 3]$.

Solution

(a) curve:

$$f(x) = x^2 + 5x - 3 \quad \text{on} \quad [-4, 5],$$
$$f'(x) = 2x + 5.$$

For critical points, $f'(x) = 0$, namely $x = -\dfrac{5}{2}$.

Therefore, at the critical point $f(-\dfrac{5}{2}) = -9\dfrac{1}{4}$.

At the endpoints $f(-4) = -7$ and $f(5) = 47$.

Hence, the global minimum value is $-9\dfrac{1}{4}$ and the global maximum value is 47.

(b) curve: $f(x) = \dfrac{x}{x^2+1}$ on $[-3, 3]$.

For critical point, $f'(x) = 0$,

$$\frac{1-x^2}{(x^2+1)^2} = 0,$$

$$1 - x^2 = 0.$$

Thus, $x = \pm 1$.

At the critical points, $f(1) = \dfrac{1}{2}$ and $f(-1) = -\dfrac{1}{2}$.

At the endpoints, $f(-3) = -\dfrac{3}{10}$ and $f(3) = \dfrac{3}{10}$.

Hence, the global minimum value is $-\dfrac{1}{2}$ and global maximum vale is $\dfrac{1}{2}$.

[illegible] local maximum and local minimum

[illegible]

(i) [illegible] then f has a local maximum at c.

(ii) [illegible] then f has a local minimum at c.

(iii) [illegible]

Example 8

Find the critical points of

$$f(x) = x^3 - 3x + 1$$

and the intervals in which the function is increasing or decreasing. Identify the local maximum and minimum.

Solution

$$f(x) = x^3 - 3x + 1,$$

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

Critical point: $f'(x) = 0$, namely $x = \pm 1$.

Interval	$x < -1$	$-1 < x < 1$	$x > 1$
[illegible]	[illegible]	[illegible]	[illegible]
sign of $f'(x)$	[illegible]	[illegible]	[illegible]
behaviour of f	increasing	decreasing	increasing

local maximum at $x = -1$ and local minimum at $x = 1$.

[illegible]

[illegible] is increasing on both sides of critical point

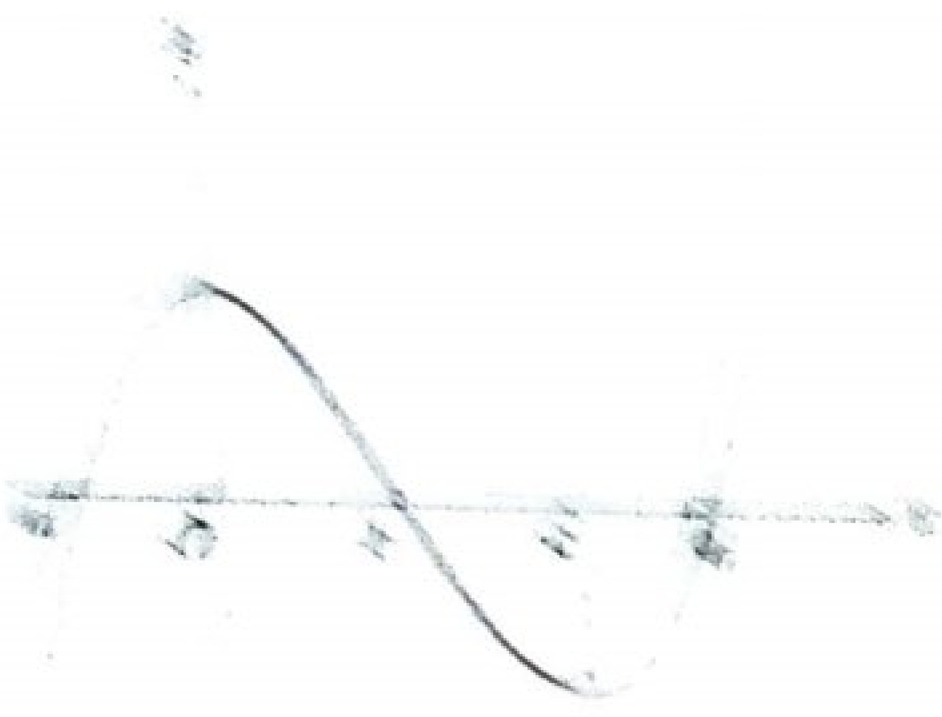
[illegible]

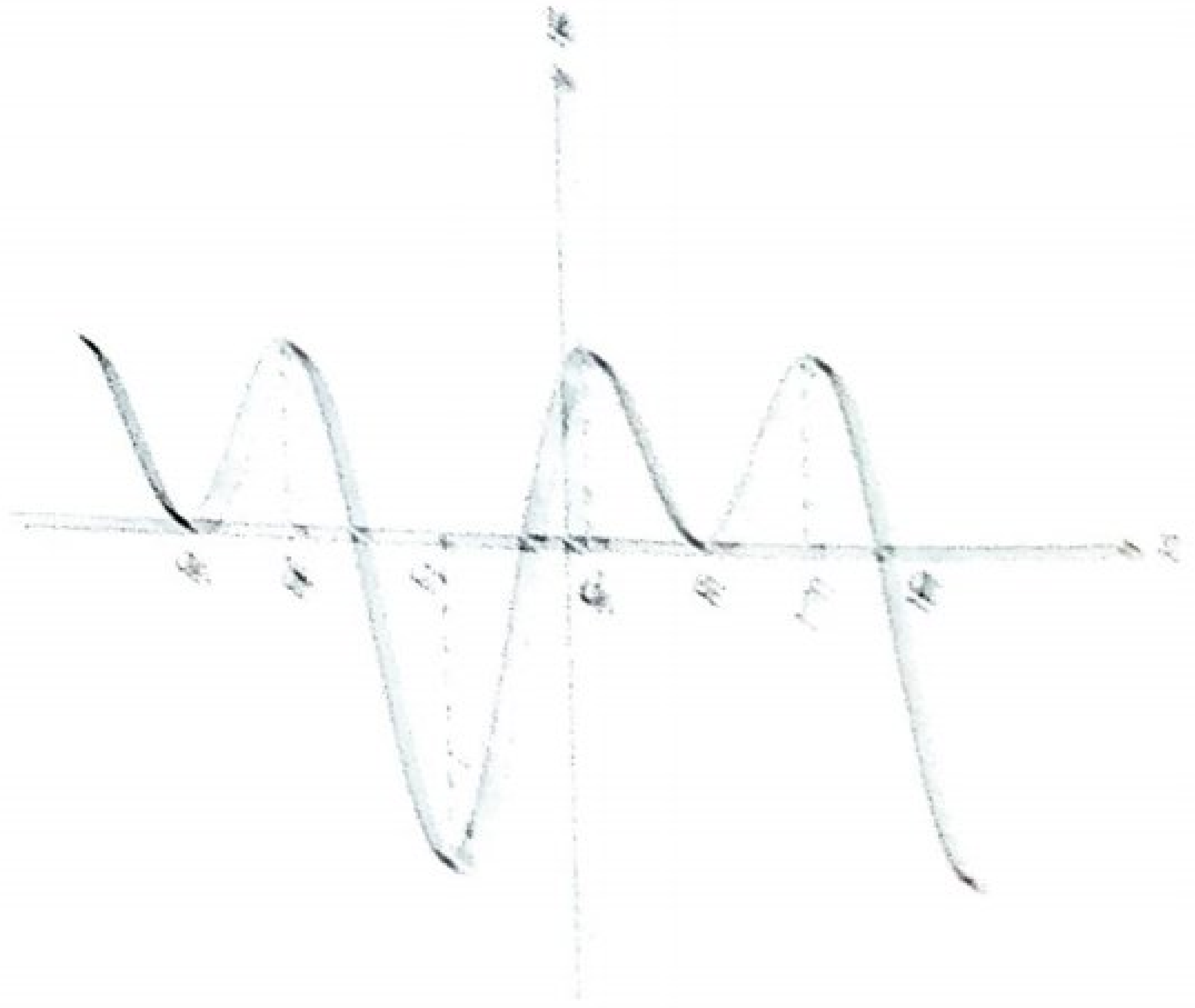
Interval	$x < 0$	$x > 0$
x^2	+	+
sign of $f'(x)$	+	+
Behavior of f	increasing	increasing

Thus the function is increasing on both sides of critical point.

Exercise 4.2

1. Consider the following graphs.





(a) State all open intervals where the function is increasing.

(b) State all open intervals where the function is decreasing.

2. Find the critical points.

(a) $f(x) = x^2 - 4x + 3$ (b) $f(x) = x^3 + 3x^2 - 9x + 10$ (c) $f(x) =$ [illegible]

(d) $f(x) = (x^2 - 4x)^{\frac{2}{3}}$ (e) $f(x) = \frac{1}{x^2 - 4x + 3}$ (f) $f(x) =$ [illegible]

3. Find the global minimum and global maximum values of the following functions

(a) $f(x) = 2x - x^2 + 2$ on $[-1, 3]$ (b) $f(x) = \sin 2x$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

4. Find the open intervals on which the following functions are increasing or decreasing. Identify the function's local maximum and minimum.

(a) $f(x) = x^3 - 3x^2 + 3$ (b) $f(x) = (x^2 - 4x)^{\frac{2}{3}}$ (c) $f(x) =$ [illegible]

(d) $f(x) = -x^3 - 3x^2 + 12$ (e) $f(x) = \frac{x}{2} - \frac{2}{x}$ (f) $f(x) =$ [illegible]

Concavity

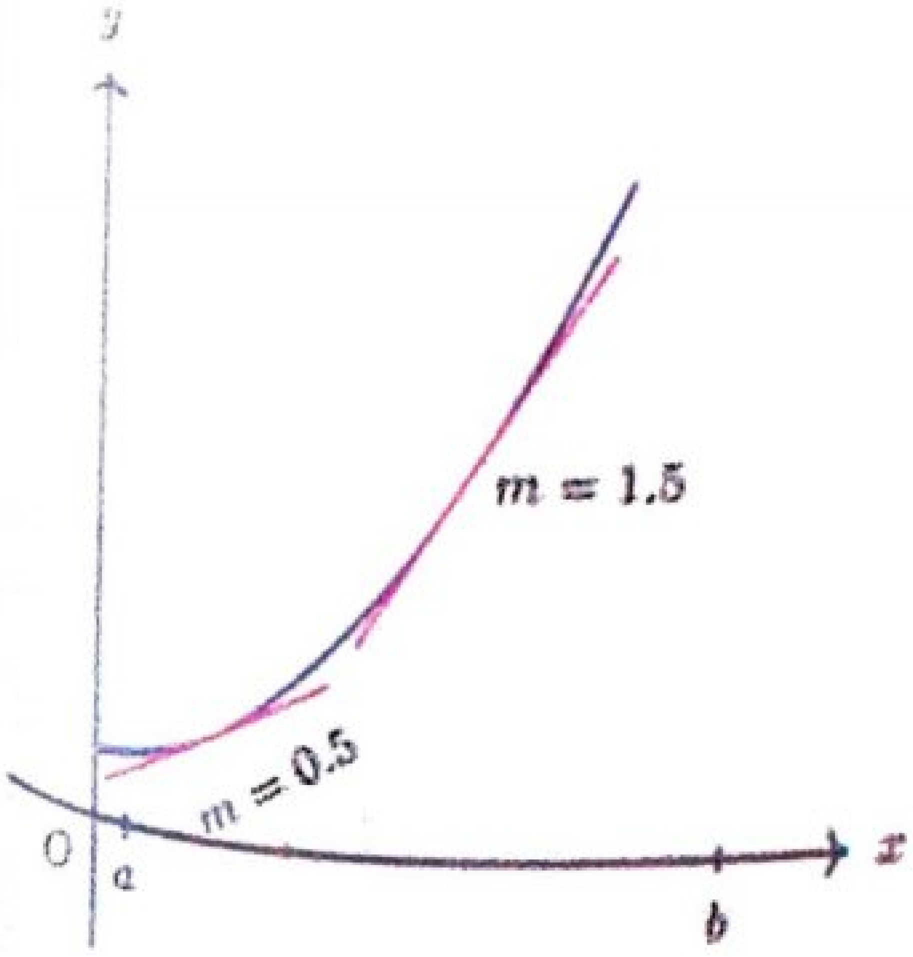
We now introduce the concavity of the graph of functions and then define the point of inflection. Next, we describe the second derivative test to find the local maximum and local minimum.

The nature of $f'(x)$ can determine when the graph of a function is bending up or down. More precisely, when $f'(x)$ is increasing on (a, b), the slope of the tangent line to the curve increases, and the graph bends upward. Similarly, when $f'(x)$ is decreasing on (a, b), the slope of the tangent line to the curve decreases, and the graph bends downward.

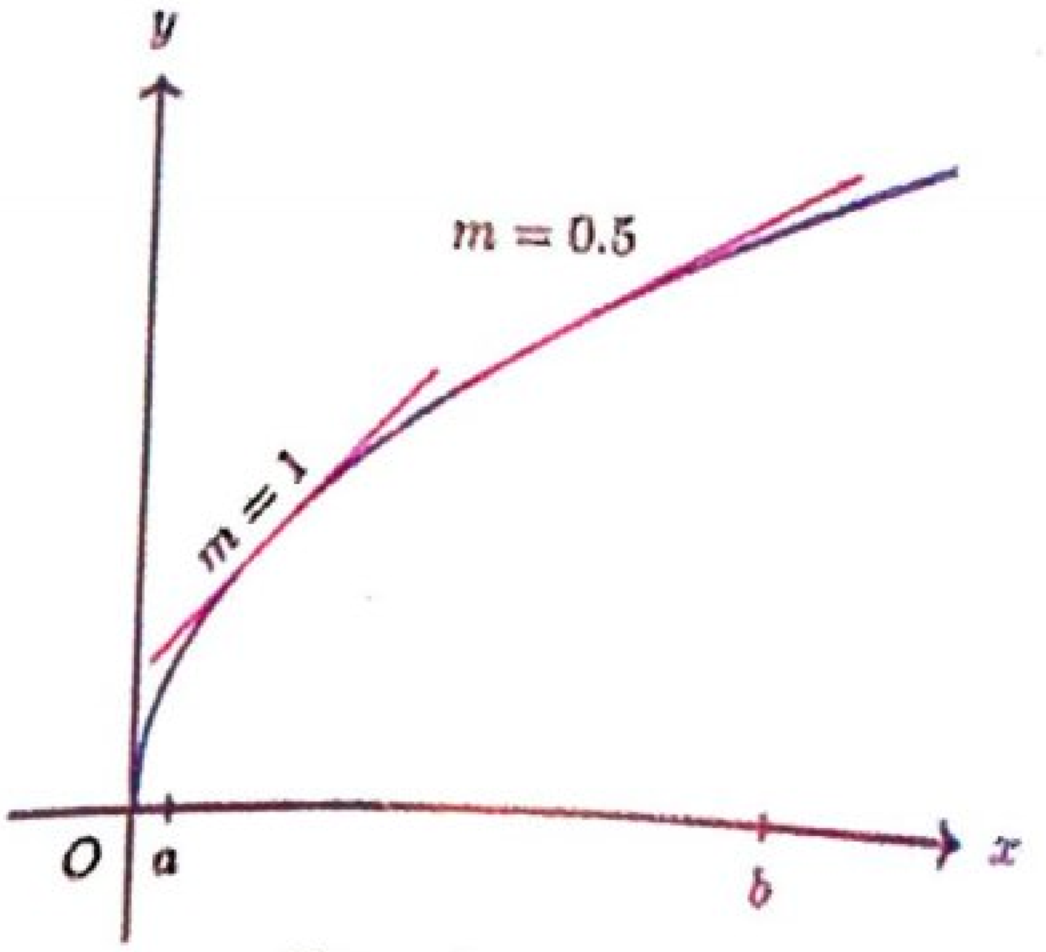
Definition.

Let f be differentiable on an open interval I. The graph of f is

1. concave **up** on I if f' is increasing on the interval.
2. concave **down** on I if f' is decreasing on the interval.



Slopes increasing



Slopes decreasing

If $f''(x) > 0$ for all x in the interval (a, b), then $f'(x)$ is increasing on (a, b). So the graph of f is concave up on (a, b).

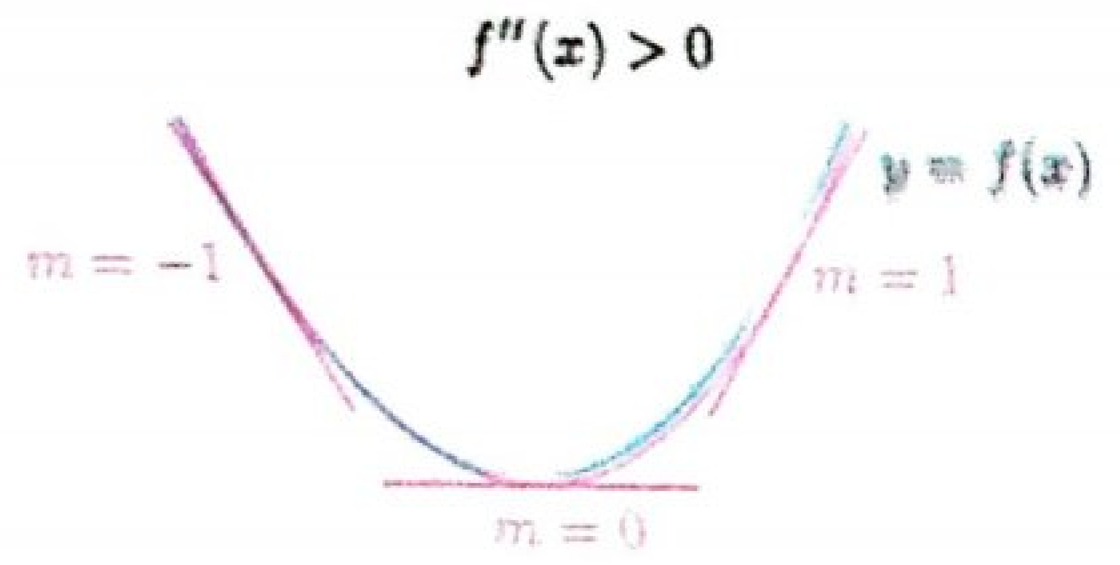
Again, if $f''(x) < 0$ for all x in the interval (a, b), then $f'(x)$ is decreasing on (a, b). So the graph of f is concave down on (a, b).

Therefore we can use the second derivative to decide whether the function is concave up or concave down.

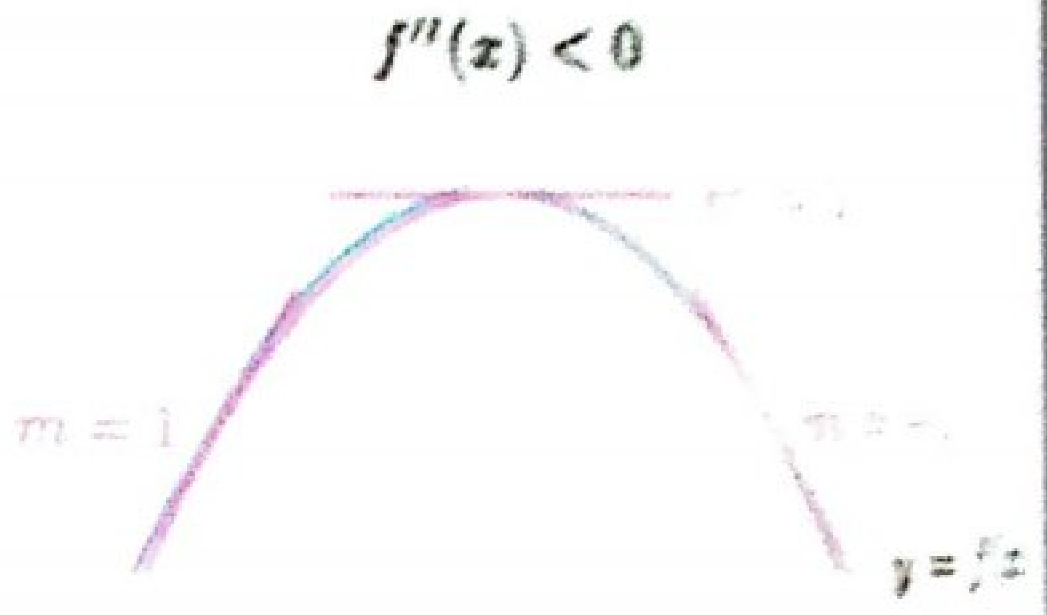
Test for Concavity

Let f be twice differentiable on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is concave up on I.
2. If $f''(x) < 0$ for all x in I, then f is concave down on I.



Concave up



Concave down

Example 11.

Determine the open intervals on which the graph of the function $f(x) = \frac{1}{x^2 +}$ concave up or concave down

Solution

$$f(x) = \frac{1}{x^2+1},$$

$$f'(x) = -\frac{2x}{(x^2+1)^2}.$$

We calculate that

$$\begin{aligned} f''(x) &= \frac{-2(x^2+1)^2 + 2x(2)(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} = \frac{2(3x^2-1)}{(x^2+1)^3}. \end{aligned}$$

$f''(x)$ is defined for all real numbers and $f''(x) = 0$.

Hence,

$$\begin{aligned} \frac{2(3x^2-1)}{(x^2+1)^3} &= 0 \\ 2(3x^2-1) &= 0 \\ (x + \frac{1}{\sqrt{3}})(x - \frac{1}{\sqrt{3}}) &= 0. \end{aligned}$$

erefore $x = \pm\frac{1}{\sqrt{3}}$.

Interval	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$x > \frac{1}{\sqrt{3}}$
$x + \frac{1}{\sqrt{3}}$	$-$	$+$	$+$
$x - \frac{1}{\sqrt{3}}$	$-$	$-$	$+$
sign of $f''(x)$	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We now describe a behavior of a point at which the concavity changes.

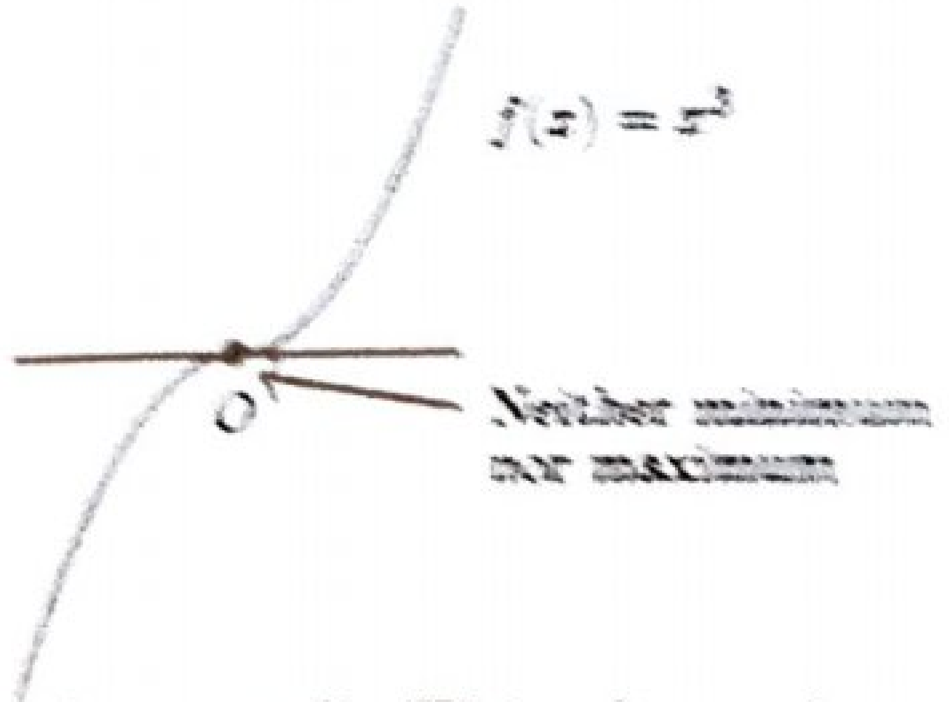
Consider the function $f(x) = x^3$.

For critical point,

$$f'(x) = 0,$$

$$3x^2 = 0.$$

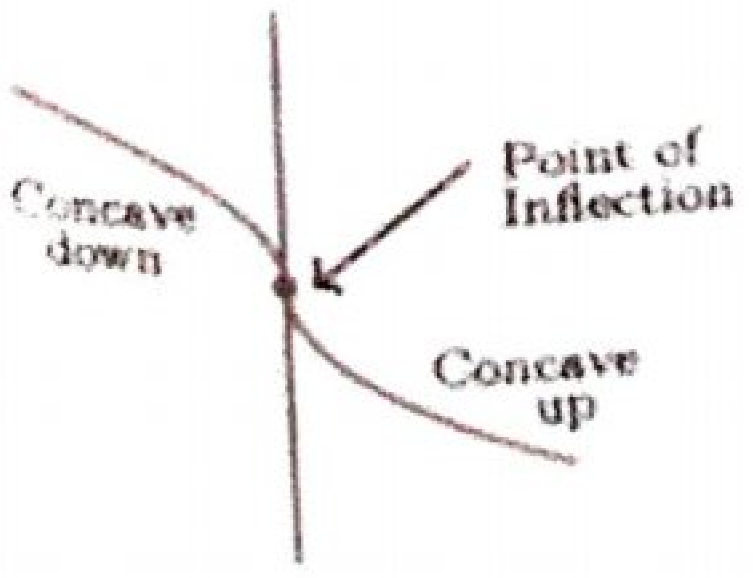
$$\text{Thus} \quad x = 0.$$

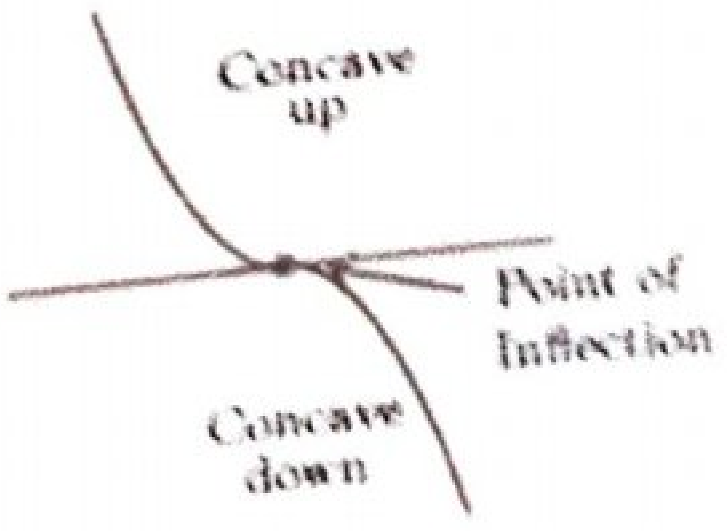

$f(x) = x^3$
O
Neither minimum nor maximum

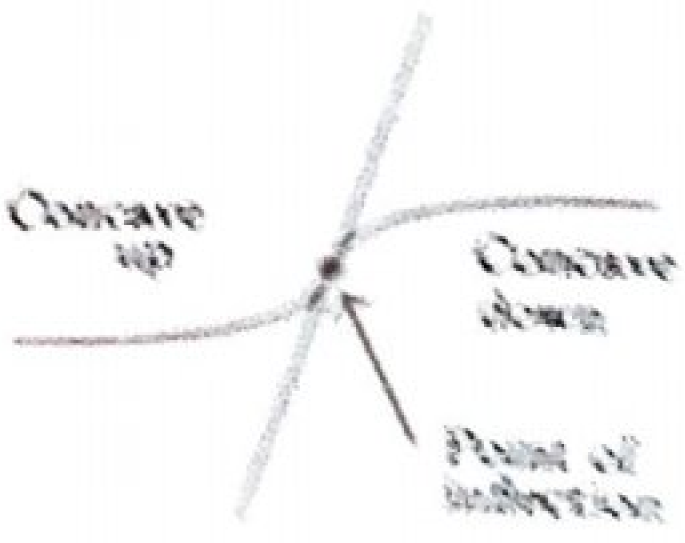
Now $f''(x) = 6x$. So $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. This show that the concavity changes at $(0,0)$ and the tangent to this curve crosses the curve at $(0,0)$.

Definition.

A point where the graph of a function has a tangent line and where the concavity changes is **a point of inflection**. At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ is undefined.


Concave down
Point of Inflection
Concave up


Concave up
Point of Inflection
Concave down


Concave up
Concave down
Point of inflection

Remark. We have an exceptional case that a point c is not inflection point even if $f''(c) = 0$. For example,

$$f(x) = (x - c)^{2k}$$

where the integer $k \geq 2$.

Example 12.

Determine the open intervals on which the graph of following functions

(a) $f(x) = x^{\frac{1}{3}}$ (b) $f(x) = \dfrac{9}{88}x^{\frac{11}{3}}$

is concave up or concave down. State their inflection points.

Solution

(a)

$$f(x) = x^{\frac{1}{3}},$$
$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}},$$
$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9x^{\frac{5}{3}}}.$$

Interval	$x < 0$	$x = 0$	$x > 0$
sign of $f''(x)$	$+$	undefined	$-$
Behavior of f	concave up	inflection point	concave down

Therefore on $x < 0$ the graph of function is concave up and on $x > 0$ the graph of function is concave down. The point $(0, 0)$ is an inflection point.

(b)

$$f(x) = \frac{9}{88}x^{\frac{11}{3}},$$
$$f'(x) = \frac{3}{8}x^{\frac{8}{3}},$$
$$f''(x) = x^{\frac{5}{3}}.$$

Interval	$x < 0$	$x = 0$	$x > 0$
sign of $f''(x)$	$-$	$f''(0) = 0$	$+$
Behavior of f	concave down	inflection point	concave up

Therefore on $x < 0$ the graph of function is concave down and on $x > 0$ the graph function is concave up. The point $(0, 0)$ is an inflection point.

The following example shows that concavity of the curve does not change at $(0, 0)$ if $f''(0) = 0$.

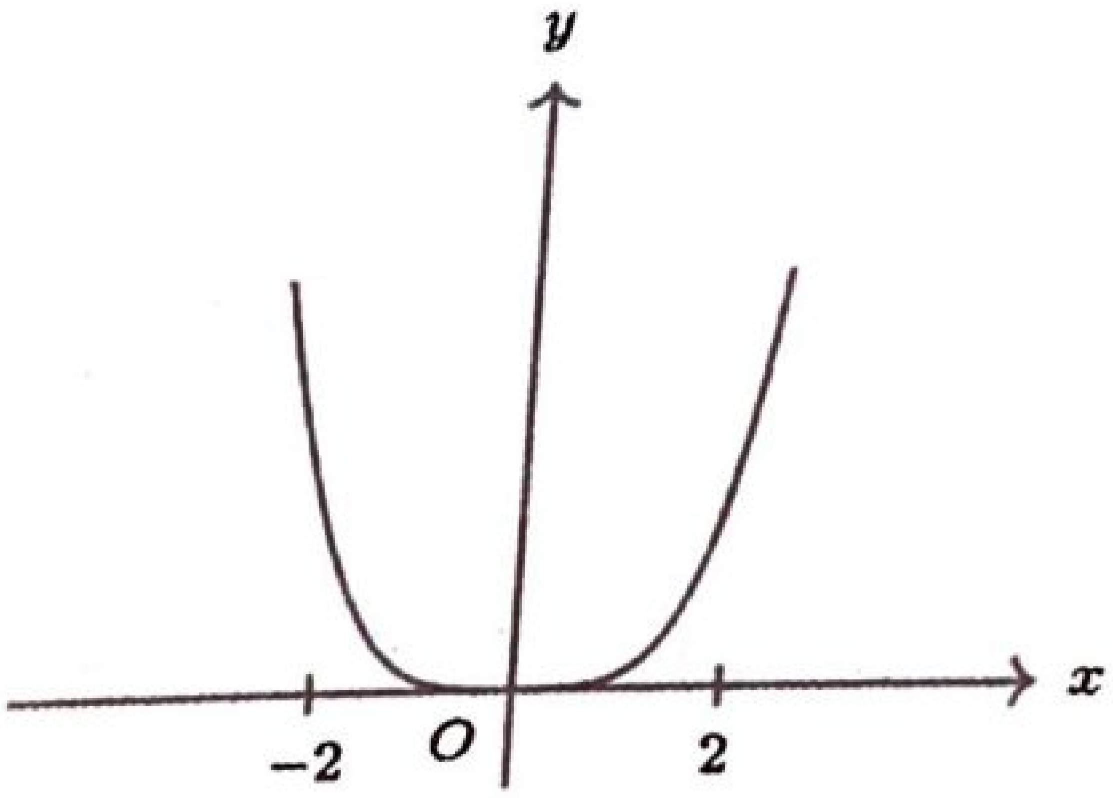
Example 13.

Show that $x = 0$ is not inflection point of $f(x) = x^6$.

Solution

curve: $f(x) = x^6$,

$$f'(x) = 6x^5.$$



For critical points,

$f'(x) = 0$, namely $x = 0$.

Next, we consider $f''(x) = 30x^4$.

At $x = 0$, $f''(0) = 0$.

We observe that

$$f''(x) > 0 \text{ for } x > 0, \text{ and } f''(x) > 0 \text{ for } x < 0,$$

so the second derivative dose not change sign at $x = 0$.

Therefore, $x = 0$ is not inflection point of $f(x) = x^6$.

Second Derivative Test

We now discuss the use of second derivative to determine whether a function is maximum or minimum without looking for sign changes in f' at critical points (see Figure in page 158).

Second Derivative Test for Local Maximum and Local Minimum

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example 14.

Find and classify the critical points of $f(x) = x^4 - 4x^3 + 5$.

Solution

curve:
$$f(x) = x^4 - 4x^3 + 5,$$
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3).$$

For critical points, $f'(x) = 0$, namely $x = 0$ and $x = 3$.

When $x = 0$, $f(0) = 5$, and when $x = 3$, $f(3) = -22$.

Therefore, the critical points are $(0, 5)$ and $(3, -22)$.

Next, $f''(x) = 12x^2 - 24x = 12x(x-2)$.

At $x = 0$, $f''(0) = 0$. The second derivative test fails.

It is observed that $f'(x) < 0$ when $x < 0$ and $f'(x) < 0$ when $0 < x < 3$.

Therefore $(0, 5)$ is a point of inflection.

At $x = 3$, $f''(3) = 36$.

Therefore $(3, -22)$ is a local minimum.

Example 15.

Find the range of $f(x) = \ln x - x$.

Solution

curve:
$$f(x) = \ln x - x,$$
$$f'(x) = \frac{1}{x} - 1,$$
$$f''(x) = -\frac{1}{x^2}.$$

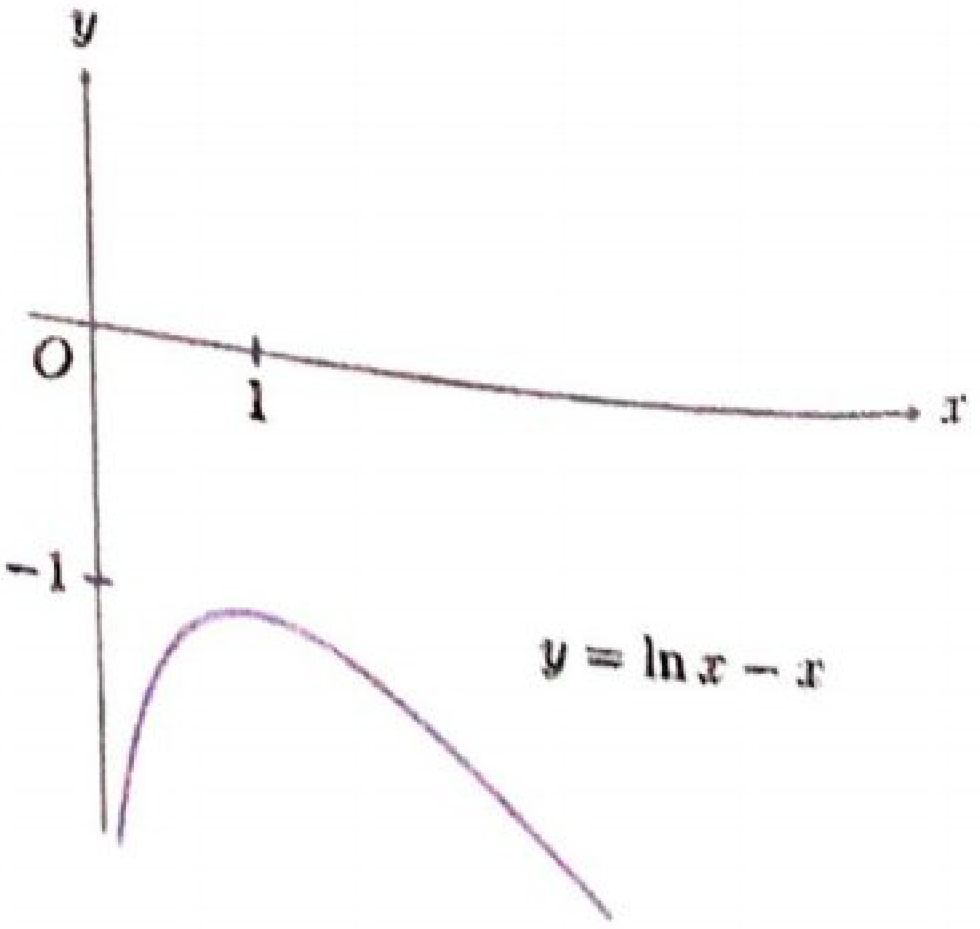
For critical point, $f'(x) = 0$,
$$\frac{1}{x} - 1 = 0.$$

Therefore $x = 1$,
and
$$f(1) = \ln 1 - 1 = -1.$$

At $x = 1$, $f''(1) = -1 < 0$.

Thus $(1, -1)$ is a maximum point.

The range of $f(x)$ is $\{y : y \leq -1\}$.



xample 16.

Find the least amount of material needed to build an open cylindrical vessel with a capacity of 400π cm^3.

Solution

Let r = radius and h = height of cylindrical vessel. Then volume of vessel is

$$\pi r^2 h = 400\pi$$
$$h = \frac{400}{r^2}.$$

Since vessel is open top, area of material for vessel is

$$A = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r \frac{400}{r^2}$$
$$= \pi r^2 + \frac{800\pi}{r}, \ (r > 0).$$

Differentiating both sides with respect to r, we get

$$\frac{dA}{dr} = 2\pi r - \frac{800\pi}{r^2}.$$

For critical point, $\frac{dA}{dr} = 0$

$$2\pi r - \frac{800\pi}{r^2} = 0$$
$$r^3 = 400$$
$$r = \sqrt[3]{400}.$$

Next, $\frac{d^2A}{dr^2} = 2\pi + \frac{1600\pi}{r^3}$.

When $r = \sqrt[3]{400}$, $\frac{d^2A}{dr^2} = 2\pi + 4\pi > 0$.

Therefore the area of material is the least when $r = \sqrt[3]{400}$.

Therefore the least amount of material $= \pi r^2 + \frac{800\pi}{r} = \frac{1200\pi}{\sqrt[3]{400}}$ cm^3.

Exercise 9.3

1. Determine the open intervals on which the graph of the following functions are concave up or concave down. State their points of inflection.

   (a) $f(x) = (x-1)^5$ (b) $g(x) = x^4 - 2x^3$ (c) $h(x) = \sin x$ on $[-\pi, \pi]$

2. Find and classify the critical points of the following functions.

   (a) $f(x) = x^4 - 4x^3 + 5$ (b) $g(x) = x^5 - 5x + 5$ (c) $h(x) = xe^{-x}$

3. Find the range of $f(x) = x - e^x$.

4. If a piece of string of fixed length is made to enclose a rectangle, show that the enclosed area is the greatest when the rectangle is a square.

5. Find the minimum value of the sum of a positive number and its reciprocal.

6. A rectangular field is surrounded by a fence on three of its sides and a straight hedge on the fourth side. If the length of the fence is 320 meters, find the maximum area of the field enclosed.

7. A rectangular box has a square base of side x cm. If the sum of one side of the square and height is 15 cm, express the volume of the box in terms of x. Use this expression to determine the maximum volume of the box.

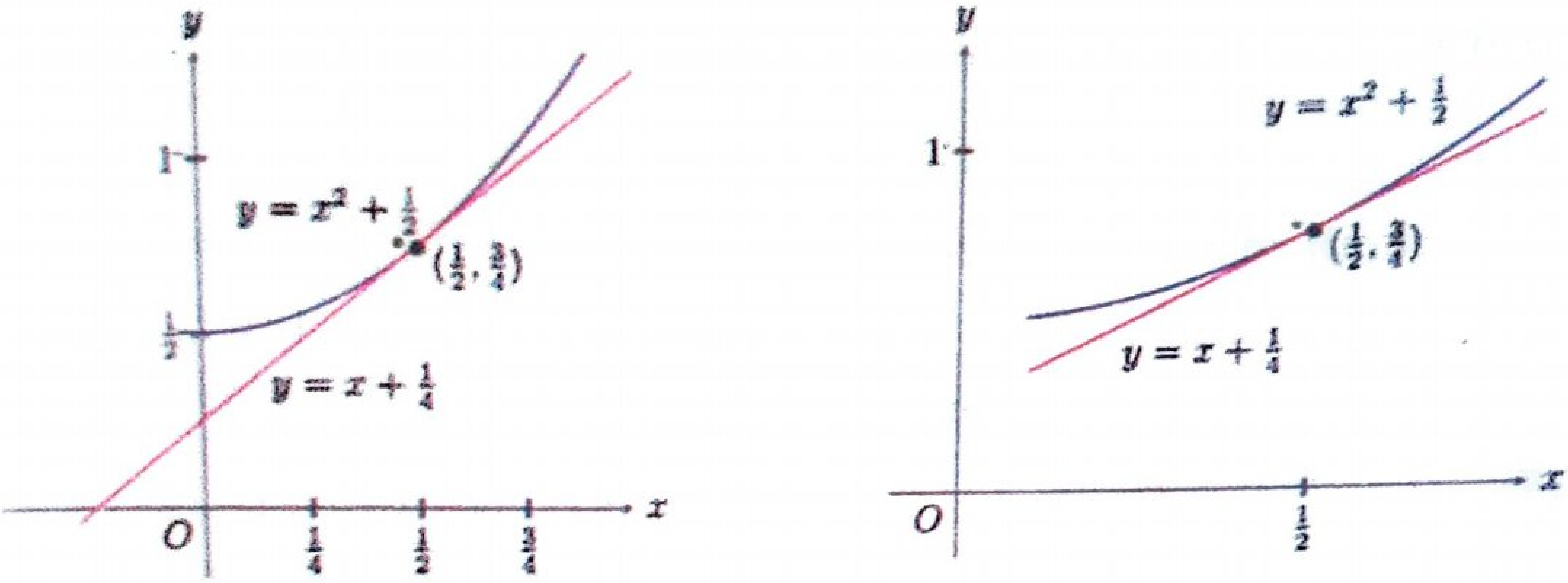
8. Two sides of triangle have lengths x and y, and the angle between them is θ. What value of θ will maximize the triangle's area?

9.3 Linearization

When we have a complicated function that can be impossible to make a perfectly accurate value, we use the linearization that are based on its tangent lines.

Consider the tangent to the curve $y = f(x)$ that lies closed to the curve near the point of tangency. For a small interval to either side of the point of tangency, the y-values along the tangent line give good approximations to the y-values of the curve.

For example, we can see this phenomenon by zooming in on the graph of $y = x^2 + \frac{1}{2}$.



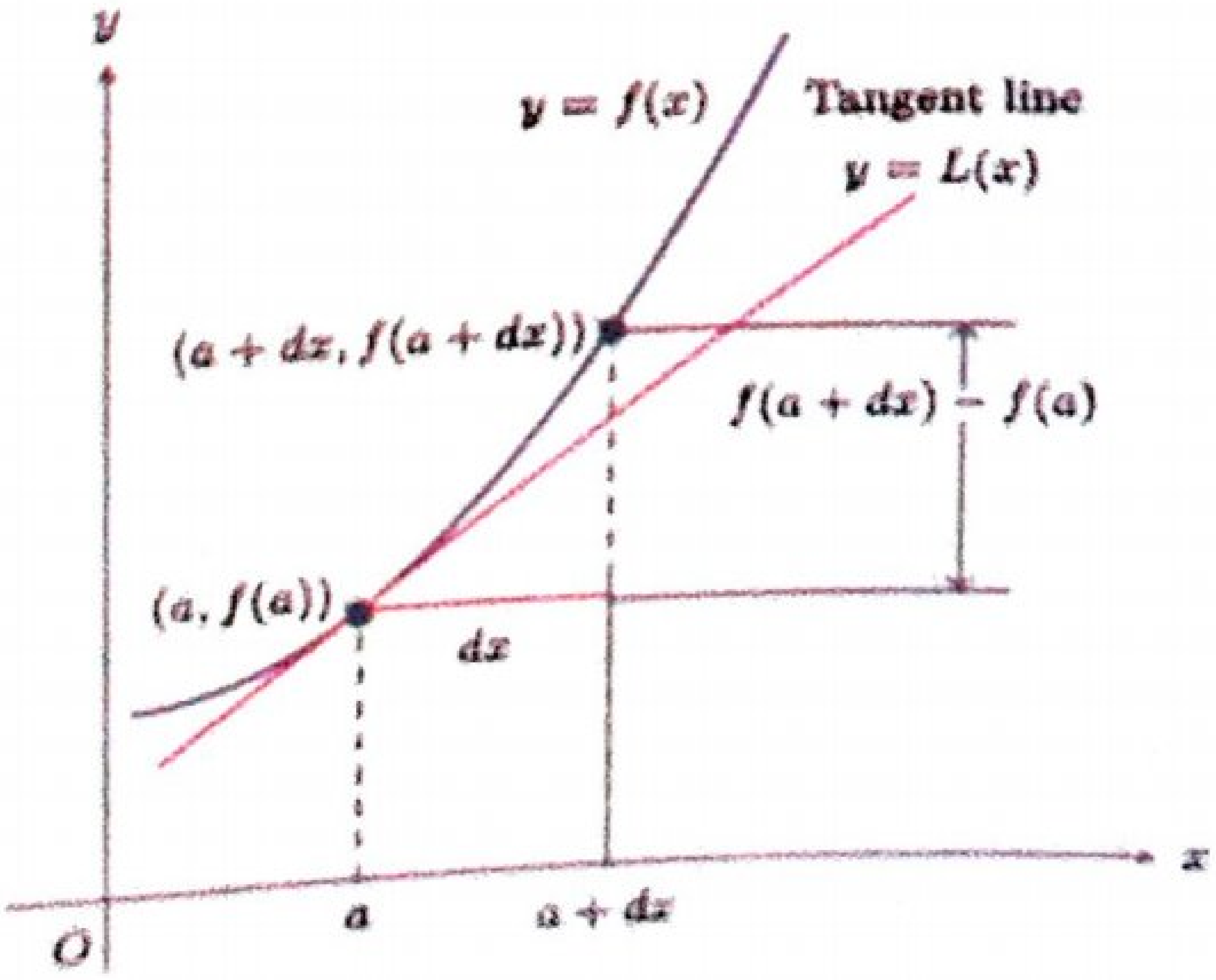
Curve $y = x^2 + \dfrac{1}{2}$ and its tangent $y = x + \dfrac{1}{4}$ are very close near $(\dfrac{1}{2}, \dfrac{3}{4})$

The equation of the tangent to the curve $y = f(x)$ at $x = a$, where f is differentiable and passes through the point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

This tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$



For as long as $L(x)$ is close to the graph of f, the function $L(x)$ gives a good approximation to $f(x)$.

Definition.

If f is differentiable at $x = a$, then the approximation function

$$L(x) = f(a) + f'(a)(x - a)$$

he **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

f by L is the **standard linear approximation** of f at a. The point $x = a$ is the enter of the approximation.

he difference of $f(x)$ and $L(x)$ is the error of the linearization.

xample 17.

ind the linearization of $f(x) = x^2 + 2x$ at $x = 2$. Compare the approximate value and true value at $x = 2.5$, $x = 2.05$ and $x = 2.005$.

Solution

$$f(x) = x^2 + 2x,$$
$$f'(x) = 2x + 2.$$

Since $f(2) = 8$ and $f'(2) = 6$, the linearization is

$$L(x) = f(2) + f'(2)(x-2) = 8 + 6(x-2) = 6x - 4.$$

That is, $x^2 + 2x \approx 6x - 4$.

At $x = 2.5$, the true value gives $x^2 + 2x = (2.5)^2 + 2(2.5) = 11.25$ and

the linearization gives $6x - 4 = 6(2.5) - 4 = 11$.

Therefore, the approximate value differs from the true value by less than 0.25.

At $x = 2.05$, the true value gives $x^2 + 2x = (2.05)^2 + 2(2.05) = 8.3025$ and

the linearization gives $6x - 4 = 6(2.05) - 4 = 8.3$.

Therefore, the approximate value differs from the true value by less than 0.25×10^{-2}.

At $x = 2.005$, the true value gives $x^2 + 2x = (2.005)^2 + 2(2.005) = 8.030025$ and

the linearization gives $6x - 4 = 6(2.005) - 4 = 8.03$.

Therefore, the approximate value differs from the true value by less than 0.25×10^{-4}.

Note: From the above example, we see that how accurate the approximation is for some values of x near 2. It turns out that, if the variable x moves a small increment then the true value and approximate value become close.

Example 18.

If the radius of a circle increases from $r = 5$ cm to 5.01 cm, find the approximate increase in the area. Estimate the area of the enlarge circle and find the error.

Solution

Let A be the area of the circle of radius r.

$$A(r) = \pi r^2.$$

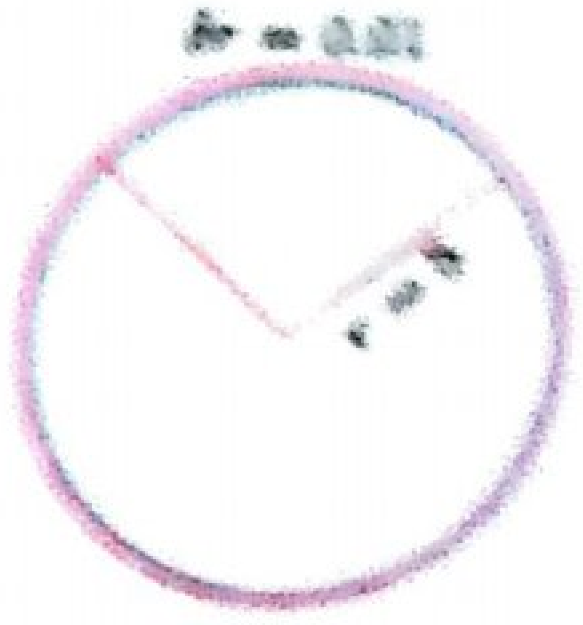
$$A'(r) = 2\pi r.$$

Since $A(5) = 25\pi$ and $A'(5) = 10\pi$, the linearization is

$$L(r) = A(5) + A'(5)(r - 5) = 10\pi r - 25\pi.$$

That is, $\pi r^2 \approx 10\pi r - 25\pi$.

At $r = 5.01$, $L(r) = L(5.01) = 10\pi(5.01) - 25\pi = 25.1\pi$.



The area of circle of radius 5.01 cm is approximately 25.1π cm². Hence the approximate increase is 0.1π cm².
The true area is

$$A(5.01) = \pi(5.01)^2 = 25.1001\pi \text{ cm}^2$$

Therefore

Error = True area − Approximate area = 0.0001π cm².

Example 19.

Given that $f(x) = x^{\frac{1}{2}}$, determine the approximate value for $\sqrt{101}$ by using approximation.

Solution

$$f(x) = x^{\frac{1}{2}}$$

When $x = 100$, $f(100) = 100^{\frac{1}{2}} = 10$, we have to approximate $\sqrt{101}$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{20}$$

The linearization is

$$L(x) = f(100) + f'(100)(x - 100) = \frac{x}{20} + 5$$

That is, $\sqrt{x} \approx \frac{x}{20} + 5$

At $x = 101$, $L(x) = L(101) = 10.05$.

Hence,

$$\sqrt{101} \approx 10.05.$$

Exercise 9.4

1. Find the linearization of
   (a) $f(x) = \sqrt{1+x}$ at $x = 3$. Compare the approximate value and true value at $x = 3.2$, $x = 3.02$ and $x = 3.002$.
   (b) $f(x) = \dfrac{x}{x+1}$ at $x = 1$. Compare the approximate value and true value at $x = 1$, $x = 1.01$ and $x = 1.001$.
2. Show that the linearization of $f(x) = (1+x)^k$ at $x = 0$ is $L(x) = 1 + kx$.
3. Find the approximation change in the volume of a sphere when its radius decreases from 5 cm to 4.97 cm. Estimate the area of the contract circle and find the error.
4. If $y = 4\sqrt{x} + 3x^2$, find the approximate change in y when x changes from 9 to 8.98.
5. If $y = 3\sqrt{72 + x^2}$, find the approximate change in y when
   (i) x increases from 3 to 3.01,
   (ii) x decreases from 3 to 2.98.
   Find their errors.
6. Use linearization to approximate the following values
   (i) $\sqrt{80}$
   (ii) $\sqrt[3]{65}$
   (iii) $\dfrac{1}{\sqrt{1.22}}$.

Chapter 10

Method of Integration

In this chapter, we first describe how we can reverse the differentiation. Then we explain the following:

- integrating the fundamental functions,
- integrating the trigonometric, exponential and logarithmic functions,
- methods of integration: substitution, integration by parts and partial fraction.

10.1 Antiderivatives

Consider

$$f'(x) = 3x^2.$$

The natural question arises, what is the function $f(x)$? Namely, what is f in terms of x? We know that differentiation decreases the power by 1, so f must contain x^3.

If $f(x) = x^3$, then $f'(x) = 3x^2$, or

if $f(x) = x^3 + 2$, then $f'(x) = 3x^2$, or

if $f(x) = x^3 - \frac{1}{2}$, then $f'(x) = 3x^2$, etc.

It means that there are many such functions of the form

$$f(x) = x^3 + C$$

where C is an arbitrary constant.

We say that

x^3 is the **antiderivative** of $3x^2$.

Next consider

$$f'(x) = x.$$

We think the same way as above, the original function f must contain x^2. However,

$$\frac{d}{dx}x^2 = 2x,$$

we see the extra factor 2. If we multiply both sides by $\frac{1}{2}$, then

$$\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x.$$

If $f(x) = \frac{1}{2}x^2 + C$ where C is an arbitrary constant, then $f'(x) = x$, so $\frac{1}{2}x^2$ is the **antiderivative** of x.

If $F(x)$ is a function where $\boldsymbol{F'(x) = f(x)}$, then the **antiderivative of $\boldsymbol{f(x)}$** is $\boldsymbol{F(x)}$.

We call the antiderivative as **integral** and write

$$\int x \, dx = \frac{1}{2}x^2 + C,$$

where C is the constant of integration.
We read this as "the integral of x with respect to x."

In general,

$$\text{if} \quad \boldsymbol{F'(x) = f(x)} \quad \text{then} \quad \boldsymbol{\int f(x)\, dx = F(x) + C}$$

where $\boldsymbol{dx}$ means that the integration is taking place with respect to the variable x. Here $f(x)$ is called the **integrand**. The variable of integration in an integral plays no essential role. It might be x, or t, or u, or anything else:

$$\int f(x)\, dx, \quad \int f(t)\, dt, \quad \int f(u)\, du, \quad \text{etc.}$$

ample 1.

d the antiderivative of

) $\sqrt{x}$ (b) $3x^5$ (c) e^{3x} (d) $\frac{1}{x^3}$.

lution

) Since $\frac{d}{dx}x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}}$ and $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = x^{\frac{1}{2}} = \sqrt{x}$,

the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x^{\frac{3}{2}}$.

) Since $\frac{d}{dx}x^6 = 6x^5$ and $\frac{d}{dx}\left(\frac{1}{2}x^6\right) = 3x^5$,

the antiderivative of $3x^5$ is $\frac{1}{2}x^6$.

) Since $\frac{d}{dx}e^{3x} = 3e^{3x}$ and $\frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$,

the antiderivative of e^{3x} is $\frac{1}{3}e^{3x}$.

) Since $\frac{d}{dx}x^{-2} = -2x^{-3}$ and $\frac{d}{dx}\left(-\frac{1}{2x^2}\right) = \frac{1}{x^3}$,

the antiderivative of $\frac{1}{x^3}$ is $-\frac{1}{2x^2}$.

e now describe the integration of fundamental functions. We know that

$$\frac{d}{dx}x^{n+1} = (n+1)x^n.$$

e reverse of this process is

$$\boxed{\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad \text{for} \quad n \neq -1.}$$

hen $n = 0$, it is seen that

$$\int x^0\,dx = \int 1\,dx = \int dx = x + C.$$

$$\boxed{\int dx = x + C.}$$

When $n = -1$, let us consider the differentiation of $\ln x$.

Since $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, it follows that $\displaystyle\int \frac{1}{x}\,dx = \ln x + C$, for $x > 0$.

For $x < 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(-x) = \dfrac{1}{-x}(-1) = \dfrac{1}{x}$, by Chain Rule.

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0.}$$

Rules of Integration

Suppose $f(x)$ and $g(x)$ are continuous functions and $k \in \mathbb{R}$.

1. $\displaystyle\int k\,dx = kx + C.$
2. $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx.$
3. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$

Example 2.

Evaluate each of the following integrals.

(a) $\displaystyle\int (4x^5 + 1)\,dx$ (b) $\displaystyle\int \left(2x^6 - \frac{1}{3}x^3 + \frac{3}{x}\right)dx$ (c) $\displaystyle\int 5x\sqrt{x}\,dx$

(d) $\displaystyle\int \frac{(x-1)^2}{\sqrt{x}}\,dx.$

Solution

(a) $\displaystyle\int (4x^5 + 1)\,dx$

$$= 4\int x^5\,dx + \int 1\,dx$$

$= \left(\dfrac{4}{6}x^6 + C_1\right) + (x + C_2)$; ($C_1$ and C_2 are constants of integration)

$$= \left(\frac{2}{3}x^6 + x\right) + C_1 + C_2$$

$= \left(\dfrac{2}{3}x^6 + x\right) + C$. ($C = C_1 + C_2$ is another constant of integration) now on, we shall write constant of integration only in the answer.

$$\int \left(2x^6 - \frac{1}{3}x^3 + \frac{3}{x}\right) dx = 2\int x^6\, dx - \frac{1}{3}\int x^3\, dx + 3\int \frac{1}{x}\, dx$$

$$= \frac{2}{7}x^7 - \frac{1}{12}x^4 + 3\ln|x| + C.$$

$$\int 5x\sqrt{x}\, dx = 5\int x\, x^{\frac{1}{2}}\, dx = 5\int x^{\frac{3}{2}}\, dx$$

$$= 5\left(\frac{2}{5}x^{\frac{5}{2}}\right) + C$$

$$= 2x^{\frac{5}{2}} + C.$$

$$\int \frac{(x-1)^2}{\sqrt{x}}\, dx = \int \frac{x^2 - 2x + 1}{x^{\frac{1}{2}}}\, dx$$

$$= \int \left(\frac{x^2}{x^{\frac{1}{2}}} - \frac{2x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) dx$$

$$= \int x^{\frac{3}{2}}\, dx - 2\int x^{\frac{1}{2}}\, dx + \int x^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C.$$

tegrating Exponential Functions

$\frac{d}{dx}a^x = a^x \ln a,$ $\int a^x\, dx = \frac{1}{\ln a}a^x + C,$ where $a > 0,\ a \neq 1.$

When $a = e,$

$\frac{d}{dx}e^x = e^x,$ $\int e^x\, dx = e^x + C.$

ample 3.

the following integrals.

$\int \frac{e^{-x}+1}{e^{-x}}\, dx$ (b) $\int (e^x + 2x)\, dx$ (c) $\int \left(e^x \log 5 + \frac{1}{x^2}\right) dx$ (d) $\int 3^x \ln 3\, dx.$

tion

$$\int \frac{e^{-x}+1}{e^{-x}}\, dx = \int (1 + e^x)\, dx = \int 1\, dx + \int e^x\, dx = x + e^x + C.$$

(b) $$\int (e^x + 2x)\,dx = \int e^x\,dx + \int 2x\,dx = e^x + x^2 + C.$$

(c) $$\int (e^x \log 5 + \frac{1}{x^2})\,dx = \int e^x \log 5\,dx + \int x^{-2}\,dx = e^x \log 5 - \frac{1}{x} + C.$$

(d) $$\int 3^x \ln 3\,dx = \frac{1}{\ln 3} 3^x\ \ln 3 + C = 3^x + C.$$

Integrating Trigonometric Functions

$\frac{d}{dx}\sin x = \cos x,$	$\int \cos x\,dx = \sin x + C.$
$\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x,$	$\int \sin x\,dx = -\cos x + C.$
$\frac{d}{dx}\tan x = \sec^2 x,$	$\int \sec^2 x\,dx = \tan x + C.$

Example 4.

Evaluate.

(a) $\int (3\cos x - 5\sin x)\,dx$

(b) $\int (e^x + 2\sin x)\,dx$

(c) $\int \frac{e^x - \sqrt{x}}{2}\,dx$

(d) $\int (1 + \tan^2 x)\,dx$

Solution

(a) $$\int (3\cos x - 5\sin x)\,dx = 3\int \cos x\,dx - 5\int \sin x\,dx = 3\sin x + 5\cos x + C$$

(b) $$\int (e^x + 2\sin x)\,dx = \int e^x\,dx + 2\int \sin x\,dx = e^x - 2\cos x + C.$$

(c) $$\int \frac{e^x - \sqrt{x}}{2}\,dx = \frac{1}{2}\int e^x\,dx - \frac{1}{2}\int x^{\frac{1}{2}}\,dx = \frac{1}{2}e^x - \frac{1}{3}x^{\frac{3}{2}} + C.$$

(d) $$\int (1 + \tan^2 x)\,dx = \int \sec^2 x\,dx = \tan x + C.$$

Integrating $f(ax+b)$

We have learned the reverse process of differentiation as an integration. Now we consider the integral of function which used the chain rule for differentiation. Consider the integral

$$\int (4x+1)^5\,dx.$$

By the chain rule,

$$\frac{d}{dx}(4x+1)^6 = 6(4x+1)^5\frac{d}{dx}(4x+1) = 24(4x+1)^5.$$

Multiplying by $\frac{1}{24}$ on both sides, we get

$$\frac{1}{24}\frac{d}{dx}(4x+1)^6 = (4x+1)^5,$$

$$\frac{d}{dx}\left(\frac{1}{24}(4x+1)^6\right) = (4x+1)^5.$$

Therefore

$$\int (4x+1)^5\,dx = \frac{1}{24}(4x+1)^6 + C.$$

For $n \neq -1$,

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = (ax+b)^n.$$

$$\boxed{\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + C, \quad n \neq -1.}$$

When $n = -1$,

$$\frac{d}{dx}\left(\frac{1}{a}\ln(ax+b)\right) = \frac{1}{a}\left(\frac{a}{ax+b}\right) = \frac{1}{ax+b}, \qquad a \neq 0.$$

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.}$$

In general, if f is a differentiable function of x, then

$$\frac{d}{dx}f(ax+b) = f'(ax+b)\frac{d}{dx}(ax+b) = af'(ax+b)$$

and reversing this we get

$$\boxed{\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b)+C.}$$

For the trigonometric functions of the form $f(ax+b)$.

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C.$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C.$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C.$$

For the exponential functions of the form $f(px+q)$,

$$\int a^{px+q}\,dx = \frac{1}{p}\frac{1}{\ln a}a^{px+q}+C \quad \text{where } a>0,\ a\neq 1.$$

When $a=e$,

$$\int e^{px+q}\,dx = \frac{1}{p}e^{px+q}+C.$$

Example 5.

Evaluate the following integrals.

(a) $\int \frac{1}{3-4x}\,dx$ (b) $\int \sqrt{8x-7}\,dx$ (c) $\int e^{2x+3}\,dx$

(d) $\int 2^{3x+1}\,dx$ (e) $\int \cos^2 x\,dx$ (f) $\int \sin 4x\cos 3x\,dx$

Solution

(a) $$\int \frac{1}{3-4x}\,dx = -\frac{1}{4}\ln|3-4x|+C$$

(b) $$\int \sqrt{8x-7}\,dx = \frac{1}{8}\left[\frac{2}{3}(8x-7)^{\frac{3}{2}}\right]+C = \frac{1}{12}(8x-7)^{\frac{3}{2}}+C.$$

(c) $\int e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + C.$

(d) $\int 2^{3x+1}\,dx = \frac{1}{3}\frac{1}{\ln 2}\,2^{3x+1} + C.$

(e) We use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, then

$$\begin{aligned}\int \cos^2 x\,dx &= \frac{1}{2}\int (1 + \cos 2x)\,dx\\ &= \frac{1}{2}(x + \tfrac{1}{2}\sin 2x) + C\\ &= \frac{1}{2}x + \tfrac{1}{4}\sin 2x + C.\end{aligned}$$

(f) We use $\sin 4x \cos 3x = \frac{1}{2}(\sin 7x + \sin x)$, then

$$\begin{aligned}\int \sin 4x \cos 3x\,dx &= \frac{1}{2}\int (\sin 7x + \sin x)\,dx\\ &= \frac{1}{2}(-\frac{1}{7}\cos 7x - \cos x) + C\\ &= -\frac{1}{14}\cos 7x - \frac{1}{2}\cos x + C.\end{aligned}$$

Exercise 10.1

1. Evaluate the following integrals:

(a) $\int 4x^8\,dx$

(b) $\int \frac{3}{2}x^2\sqrt[3]{x}\,dx$

(c) $\int (5^x + 2)\,dx$

(d) $\int \sin^2 x\,dx$

(e) $\int \frac{x+3}{\sqrt{x}}\,dx$

(f) $\int \left(\frac{1}{2x} + 5\right)\,dx$

(g) $\int \left(e^x + \frac{2}{x}\right)\,dx$

(h) $\int \left(\frac{1}{x^3} + 4e^x\right)\,dx$

(i) $\int \left(\frac{3}{x} + e^x + 10\right)\,dx$

(j) $\int \sin^2 3x\,dx$

(k) $\int \sin 5x \sin 2x\,dx$

(l) $\int \cos 7x \cos 4x\,dx$

2. Evaluate the following integrals:

(a) $\int (1-2x)^3 \, dx$ (b) $\int \sin(2\pi x + 7) \, dx$ (c) $\int \cos(3x-7) \, dx$

(d) $\int 3^{5x-2} \, dx$ (e) $\int \frac{1}{7x-6} \, dx$ (f) $\int \frac{\sin 2x}{\sin x} \, dx$

(g) $\int \sec^2(2x+3) \, dx$ (h) $\int e^{7x-3} \, dx$. (i) $\int (1+\tan^2 2\pi x) \, dx$

10.2 Substitution Method

We use the substitution method when the given integral can be transformed to the simpler integral by a change of variable. We describe how to get the integrals $\int f(g(x))g'(x) \, dx$ and $\int \frac{g'(x)}{g(x)} \, dx$ where g is differentiable and f is continuous. Then, we apply this method to the integral containing the trigonometric functions.

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable, the **differential dy** is given by

$$\boxed{dy = f'(x) \, dx.}$$

Here, dy is a dependent variable that depends on x and dx.

1. $\int f(g(x))g'(x) \, dx$

Suppose the function g is differentiable and f is continuous.

Let $u = g(x)$. Then $du = g'(x)dx$.

Substitute these into $\int f(g(x))g'(x)dx$, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u) \, du.}$$

To compute the integral by substitution method, the basic steps are as follows:

Step 1 Select a substitution $u = g(x)$.

Step 2 Differentiate the substitution and arrange to write dx in term of du.

Step 3 Substitute the expression from step 2, and transform the entire integral from x-variable to u-variable form.

Step 4 Integrate with respect to u.

Step 5 Rewrite the answer in terms of x.

Example 6.

Evaluate each of the following integrals.

(a) $\int xe^{x^2}\,dx$ (b) $\int \sin^4 x\cos x\,dx$ (c) $\int x^2\left(\frac{x^3}{3}+1\right)^3\,dx$

(d) $\int \sqrt{1-5x}\,dx$ (e) $\int 3x\sqrt{x^2+5}\,dx$ (f) $\int \cos^3 x\,dx$

Solution

(a) $\int xe^{x^2}\,dx$

Let $u = x^2$. Then $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$.

$$\int xe^{x^2}\,dx = \int e^{x^2}\,x\,dx = \int e^u\,\frac{1}{2}du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

(b) $\int \sin^4 x\cos x\,dx.$

Let $u = \sin x$. Then $du = \cos x\,dx$.

$$\int \sin x^4\cos x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

(c) $\int x^2\left(\frac{x^3}{3}+1\right)^3 dx$

Let $u = \frac{x^3}{3} + 1$. Then $du = x^2 dx$.

$$\int x^2\left(\frac{x^3}{3}+1\right)^3 dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}\left(\frac{x^3}{3}+1\right)^4 + C.$$

(d) $\int \sqrt{1-5x}\,dx$

Let $u = 1 - 5x$. Then $du = -5\,dx$, $-\frac{1}{5}\,du = dx$.

$$\begin{aligned}\int \sqrt{1-5x}\,dx = \int (1-5x)^{\frac{1}{2}}\,dx = \int u^{\frac{1}{2}}\left(-\tfrac{1}{5}\right)du &= -\frac{1}{5}\cdot\frac{2}{3}u^{\frac{3}{2}} + C\\ &= -\frac{2}{15}u^{\frac{3}{2}} + C\\ &= -\frac{2}{15}(1-5x)^{\frac{3}{2}} + C.\end{aligned}$$

(e) $\int 3x\sqrt{x^2+5}\,dx$

Let $u = x^2 + 5$. Then $\frac{1}{2}\,du = x\,dx$

$$\begin{aligned}\int 3x\sqrt{x^2+5}\,dx = 3\int (x^2+5)^{\frac{1}{2}}\,x\,dx = 3\int u^{\frac{1}{2}}\,\frac{1}{2}\,du &= \frac{3}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C\\ &= u^{\frac{3}{2}} + C\\ &= (x^2+5)^{\frac{3}{2}} + C.\end{aligned}$$

(f) $\int \cos^3 x\,dx$

We use $\cos^2 x = 1 - \sin^2 x$, then

$$\int \cos^3 x\,dx = \int (1 - \sin^2 x)\cos x\,dx.$$

et $u = \sin x$, then $du = \cos x \, dx$

$$\int \cos^3 x \, dx = \int (1 - u^2) \, du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

2. $\displaystyle\int \frac{g'(x)}{g(x)} \, dx$

et $u = g(x)$. Then $du = g'(x)dx$.

hus

$$\int \frac{g'(x)}{g(x)} \, dx = \int \frac{1}{u} \, du = \ln|u| + C.$$

$$\int \frac{g'(x)}{g(x)} \, dx = \ln|g(x)| + C.$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + C.$$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + C.$$

Example 7.

Evaluate the following integrals.

(a) $\displaystyle\int \frac{2x-1}{x^2-x-6} \, dx$ (b) $\displaystyle\int \frac{x^2+2x-1}{x^2-1} \, dx$ (c) $\displaystyle\int \frac{1}{1+e^x} \, dx$

Solution

(a) $\displaystyle\int \frac{2x-1}{x^2-x-6} \, dx$

Let $u = x^2 - x - 6$. Then $du = (2x - 1) \, dx$.

$$\int \frac{2x-1}{x^2-x-6} \, dx = \int \frac{1}{u} du = \ln|u| + C = \ln|x^2 - x - 6| + C.$$

(b) $\displaystyle\int \frac{x^2+2x-1}{x^2-1}\,dx$

We can rewrite as

$$\int \frac{x^2+2x-1}{x^2-1}\,dx = \int \left(1+\frac{2x}{x^2-1}\right)\,dx.$$

Let $u = x^2-1$. Then $du = 2x\,dx$.

$$\begin{aligned}\int \frac{x^2+2x-1}{x^2-1}\,dx &= \int \left(1+\frac{2x}{x^2-1}\right)\,dx\\ &= \int dx + \int \frac{1}{u}du\\ &= x + \ln|u| + C\\ &= x + \ln|x^2-1| + C.\end{aligned}$$

(c) $\displaystyle\int \frac{1}{1+e^x}\,dx$

We can rewrite by multiplying and dividing by e^{-x}.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+e^x}\,\frac{e^{-x}}{e^{-x}}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx.$$

Let $u = e^{-x}+1$. Then $du = -e^{-x}\,dx$.

$$\begin{aligned}\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx &= -\int \frac{1}{u}du\\ &= -\ln|u| + C\\ &= -\ln(e^{-x}+1) + C.\end{aligned}$$

We now explain some integrals involving the trigonometric functions using method of substitution.

(*a*) $\displaystyle\int \boldsymbol{\sec x\,dx = \ln|\sec x + \tan x| + C}$

Proof

We have

$$\int \sec x\,dx = \int \sec x\,\frac{\sec x+\tan x}{\sec x+\tan x}\,dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx.$$

Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x)\, dx$.

So,

$$\int \sec x \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln | \sec x + \tan x| + C.$$

(b) $$\int \csc x \, dx = - \ln | \csc x + \cot x| + C$$

Proof

We have

$$\int \csc x \, dx = \int \csc x \, \frac{\csc x + \cot x}{\csc x + \cot x} \, dx = \int \frac{\csc^2 x + \cot x \csc x}{\csc x + \cot x} \, dx.$$

Let $u = \csc x + \cot x$. Then $du = -(\cot x \csc x + \csc^2 x)\, dx$.

So,

$$\int \csc x \, dx = - \int \frac{1}{u} \, du = - \ln |u| + C = - \ln | \csc x + \cot x| + C.$$

Exercise 10.2

1. Integrate the following functions using the given substitutions.

(a) $4x^3\sqrt{x^4 - 1}$; $u = x^4 - 1$

(b) $\cos^3 x \sin x$; $u = \cos x$

(c) $\dfrac{1}{x \ln |x|}$; $u = \ln |x|$

(d) $\sin^5 x \cos x$; $u = \sin x$

(e) $\dfrac{\ln x}{x}$, $x > 0$; $u = \ln x$

(f) $x^3 e^{x^4}$; $u = x^4$

2. Use the substitution method to evaluate the following integrals.

(a) $\int x\sqrt{1 - x} \, dx$

(b) $\int (2x + 1)(x^2 + x)^7 \, dx$

(c) $\int \sin^3 x \, dx$

(d) $\int x^2 \sqrt{x^3 - 2} \, dx$

(e) $\int \dfrac{\sec^2 x}{\tan x} \, dx$

(f) $\int \dfrac{x}{\sqrt{x + 1}} \, dx$

3. Evaluate the integral $\int \dfrac{x}{(x^2 + 1) \ln(x^2 + 1)} \, dx$.

10.3 Integration by Parts

We use the method of integration by parts to integrate the product of two functions. In Section 10.1, we explain that the integral of the sum of functions is the sum of respective integrals. But the integral of the product function is not the product of respective integrals. Therefore, we use another technique and it is called the **integration by parts**.

It is based on product rule of differentiation.

$$(u\,v)' = u'\,v + u\,v'$$

where u and v are functions of x. Then integrating both sides, we get

$$\int (u\,v)'\,dx = \int u'\,v\,dx + \int u\,v'\,dx.$$

Applying the antiderivative in left hand side, we get

$$u\,v = \int u'\,v\,dx + \int u\,v'\,dx$$

or

$$\int u\,v'\,dx = u\,v - \int u'\,v\,dx.$$

Since $u'\,dx = du$ and $v'\,dx = dv$, we get

$$\boxed{\int \mathbf{u}\,d\mathbf{v} = \mathbf{u}\,\mathbf{v} - \int \mathbf{v}\,d\mathbf{u}.}$$

To compute the integral of product of two functions using integration by parts the basic steps are as follows:

Step 1 Choose u and dv.

Step 2 Differentiate u and integrate dv.

Step 3 Substitute the expression from step 2 in

$$\int u\,dv = uv - \int v\,du.$$

Step 4 Simplify.

Example 8.

Evaluate the following integrals.

(a) $\int x\, e^x\, dx$ (b) $\int \ln x\, dx$ (c) $\int x \cos x\, dx$

(d) $\int x \sin x\, dx$ (e) $\int e^x \sin x\, dx$ (f) $\int x^2 \ln x\, dx$

Solution

(a) $$\int x\, e^x\, dx$$

Let $u = x, \qquad dv = e^x\, dx.$

Then $du = dx, \qquad v = \int e^x\, dx = e^x.$

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C.$$

(b) $$\int \ln x\, dx$$

Let $u = \ln x, \qquad dv = dx.$

Then $du = \frac{1}{x} dx, \qquad v = \int dx = x.$

$$\int \ln x\, dx = x \ln x - \int x\, \frac{1}{x} dx + C = x \ln x - x + C.$$

(c) $$\int x \cos x\, dx$$

Let $u = x, \qquad dv = \cos x\, dx.$

Then $du = dx, \qquad v = \int \cos x\, dx = \sin x.$

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

(d) $$\int x \sin x\, dx$$

Let $u = x, \qquad dv = \sin x\, dx.$

Then $du = dx, \qquad v = \int \sin x\, dx = -\cos x.$

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + C.$$

(e) $\displaystyle\int e^x \sin x\,dx$

Let $\quad u = e^x, \qquad dv = \sin x\,dx.$

Then $\quad du = e^x dx, \qquad v = \displaystyle\int \sin x\,dx = -\cos x.$

$$\begin{aligned}\int e^x \sin x\,dx &= -e^x\cos x - \int(-\cos x)e^x\,dx\\ &= -e^x\cos x + \int e^x\ \cos x\,dx.\end{aligned}$$

Let $I = \displaystyle\int e^x \sin x\,dx$. Then we have

$$I = -e^x\cos x + \int e^x\ \cos x\,dx. \tag{1}$$

Let $\quad u = e^x, \qquad dv = \cos x\,dx.$

Then $\quad du = e^x dx, \qquad v = \displaystyle\int \cos x\,dx = \sin x.$

Thus

$$\int e^x \cos x\,dx = e^x\sin x - \int e^x \sin x\,dx. \tag{2}$$

Substituting (2) into (1), we get

$$\begin{aligned}I &= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx\\ &= e^x(\sin x - \cos x) - I\\ I &= \frac{1}{2}e^x(\sin x - \cos x) + C.\end{aligned}$$

Therefore

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

(f) $\displaystyle\int x^2 \ln x\,dx$

Let $\quad u = \ln x, \qquad dv = x^2\,dx.$

Then $\quad du = \dfrac{1}{x}dx, \qquad v = \displaystyle\int x^2\,dx = \frac{x^3}{3}.$

$$\begin{aligned}\int x^2 \ln x \, dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{1}{x}\, dx\\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\,\frac{x^3}{3} + C.\\ &= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.\end{aligned}$$

Exercise 10.3

1. Use the integration by parts to evaluate the following integrals.

(a) $\int s\, e^{-2s}\, ds$ (b) $\int \ln(x+1)\, dx$ (c) $\int t \sin 2t\, dt$

(d) $\int x\, 2^x\, dx$ (e) $\int x \cos 5x\, dx$ (f) $\int e^x \cos x\, dx$

10.4 Partial Fraction Method

We use the partial fraction method to integrate the rational functions.

Example 9.

Evaluate the integral $\int \frac{1}{x^2+2x-3}\, dx$.

Solution

$$\int \frac{1}{x^2+2x-3}\, dx$$

We rewrite the rational fraction as

$$\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)}.$$

However, we want this fraction as the sum of linear fractions as below.

$$\frac{1}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$$

where A and B are constants that will be determined later. Then, it can be rewritten

$$\frac{1}{x^2+2x-3} = \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}.$$

Therefore,

$$1 = A(x-1) + B(x+3)$$

$$1 = (A+B)x + (-A+3B).$$

Equating the coefficients of corresponding powers of x, we get

$$A + B = 0$$

$$-A + 3B = 1$$

which give $A = -\frac{1}{4}$ and $B = \frac{1}{4}$.

Therefore

$$\frac{1}{x^2+2x-3} = \frac{-\frac{1}{4}}{x+3} + \frac{\frac{1}{4}}{x-1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2+2x-3}\,dx &= -\frac{1}{4}\int \frac{1}{x+3}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx\\ &= -\frac{1}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C\\ &= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C.\end{aligned}$$

Example 10.

Evaluate the integral $\int \frac{1}{2x^2+3x+1}\,dx$.

Solution

$$\int \frac{1}{2x^2+3x+1}\,dx$$

First we write

$$\frac{1}{2x^2+3x+1} = \frac{1}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

and so

$$\frac{1}{2x^2+3x+1} = \frac{A(x+1) + B(2x+1)}{(2x+1)(x+1)}.$$

efore,

$$1 = A(x+1) + B(2x+1).$$

$x = -1$, $\quad 1 = B(-1)$, so $\quad B = -1$.

$x = -\frac{1}{2}$, $\quad 1 = A(\frac{1}{2})$, so $\quad A = 2$.

$$\int \frac{1}{2x^2+3x+1}\,dx = \int \frac{2}{2x+1}\,dx + \int \frac{-1}{x+1}\,dx$$

$$= \ln|2x+1| - \ln|x+1| + C$$

$$= \ln\left|\frac{2x+1}{x+1}\right| + C.$$

mple 11.

ate the integral $\int \frac{x^3+x^2-4x}{x^2-4}\,dx$.

tion

$$\int \frac{x^3+x^2-4x}{x^2-4}\,dx$$

rewrite as

$$\int \frac{x^3+x^2-4x}{x^2-4}\,dx = \int \left(x+1+\frac{4}{x^2-4}\right)\,dx.$$

write

$$\frac{4}{x^2-4} = \frac{4}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\frac{4}{x^2-4} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}.$$

fore,

$$4 = A(x-2) + B(x+2).$$

$x = 2$, $\quad 4 = B(4)$, $\quad$ so $B = 1$.

$x = -2$, $\quad 4 = A(-4)$, $\quad$ so $A = -1$.

$$\begin{aligned}\int \frac{x^3+x^2-4x}{x^2-4}\,dx &= \int (x+1)\,dx + \int \frac{-1}{x+2}\,dx + \int \frac{1}{x-2}\,dx\\ &= \frac{x^2}{2}+x-\ln|x+2|+\ln|x-2|+C\\ &= \frac{x^2}{2}+x+\ln\left|\frac{x-2}{x+2}\right|+C.\end{aligned}$$

Example 12.

Evaluate the integral $\int \frac{2x+1}{(x-1)^2}\,dx$.

Solution

$$\int \frac{2x+1}{(x-1)^2}\,dx$$

We can rewrite as

$$\frac{2x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

and so

$$\frac{2x+1}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}.$$

Therefore,

$$\begin{aligned}2x+1 &= A(x-1)+B,\\ &= Ax+(-A+B).\end{aligned}$$

Equating the coefficients of corresponding powers of x, we get

$$A=2 \quad \text{and} \quad -A+B=1$$

which give $B=3$.

Therefore

$$\frac{2x+1}{(x-1)^2} = \frac{2}{x-1} + \frac{3}{(x-1)^2}.$$

Thus

$$\int \frac{2x+1}{(x-1)^2}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{(x-1)^2}\,dx$$

$$= 2\ln|x-1| - \frac{3}{x-1} + C.$$

Example 13.

Find the function $f(x)$ satisfying the equation

$$f'(x) = \frac{x-1}{\sqrt{x}} \quad \text{with} \quad f(1) = 0.$$

Solution

$$f'(x) = \frac{x-1}{\sqrt{x}}$$

By integrating we get

$$f(x) = \int \frac{x-1}{\sqrt{x}}\,dx = \int \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$$

Since $f(1) = 0$, it follows that

$$f(1) = \frac{2}{3}(1)^{\frac{3}{2}} - 2(1)^{\frac{1}{2}} + C$$

$$0 = \frac{2}{3} - 2 + C$$

$$C = \frac{4}{3}$$

Therefore $f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + \frac{4}{3}$

Example 14.

Find the function $f(x)$ satisfying the equation

$f''(x) =$ [illegible] with $f'(0) = 1$ and $f(0) = k$.

Solution

$$f''(x) = \frac{e^x - e^{-x}}{2}$$

By integrating we get

$$f'(x) = \int \frac{e^x - e^{-x}}{2}\,dx = \frac{1}{2}(e^x + e^{-x}) + C.$$

Since $f'(0) = 1$, it follows that

$$f'(0) = \frac{1}{2}(e^0 + e^0) + C$$

$$1 = 1 + C$$

$$C = 0.$$

Therefore, $f'(x) = \frac{1}{2}(e^x + e^{-x})$.

Again, by integrating we get

$$f(x) = \int \frac{e^x + e^{-x}}{2}\,dx = \frac{1}{2}(e^x - e^{-x}) + C.$$

Since $f(0) = 1$, it follows that

$$f(0) = \tfrac{1}{2}(e^0 - e^0) + C$$

$$1 = 0 + C$$

$$C = 1.$$

Therefore, $f(x) = \frac{1}{2}(e^x + e^{-x}) + 1$.

Exercise 10.4

1. Use the partial fractions method to evaluate the following integrals.

(a) $\int \frac{1}{2x^2 + 5x + 3}\,dx$ (b) $\int \frac{2x - 1}{(x - 3)^2}\,dx$

(c) $\int \frac{x + 1}{(2x + 5)(x + 4)}\,dx$ (d) $\int \frac{2x^2 - 1}{x^2 - 1}\,dx$

2. Find the function $f(x)$ that satisfying the equation

$$f'(x) = \sin 4x \cos 2x \quad \text{with} \quad f(\frac{\pi}{2}) = 0$$

3. Find the function $g(x)$ that satisfying the equation

$$g'(x) = x^2e^{x^3} \quad \text{with} \quad g(0) = -\frac{2}{3}.$$

4. Find the function $h(x)$ that satisfying the equation

$$h'(x) = \frac{x}{x^2-1} \quad \text{with} \quad h(2) = \frac{1}{2}.$$

5. Find the function $f(x)$ that satisfying the equation

$$f''(x) = 2x - 1 \quad \text{with} \quad f(0) = -1 \quad \text{and} \quad f'(1) = 2.$$

6. Find the function $g(x)$ that satisfying the equation

$$g''(x) = x\,\sin x \quad \text{with} \quad g(\tfrac{\pi}{2}) = 0 \quad \text{and} \quad g'(0) = 0.$$

Rules for Definite Integration

Suppose f and g are continuous on $[a,b]$ and $k \in \mathbb{R}$.

1. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx.$
2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
3. $\displaystyle\int_a^a f(x)\,dx = 0.$
4. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $c \in (a,b)$.

Example 1.

Evaluate the following definite integrals.

(a) $\displaystyle\int_{-1}^{2} x^3\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$ (c) $\displaystyle\int_{-1}^{1} \sqrt{x+2}\,dx$ (d) $\displaystyle\int_{\frac{\pi}{4}}^{\pi} |\cos x|\,dx$

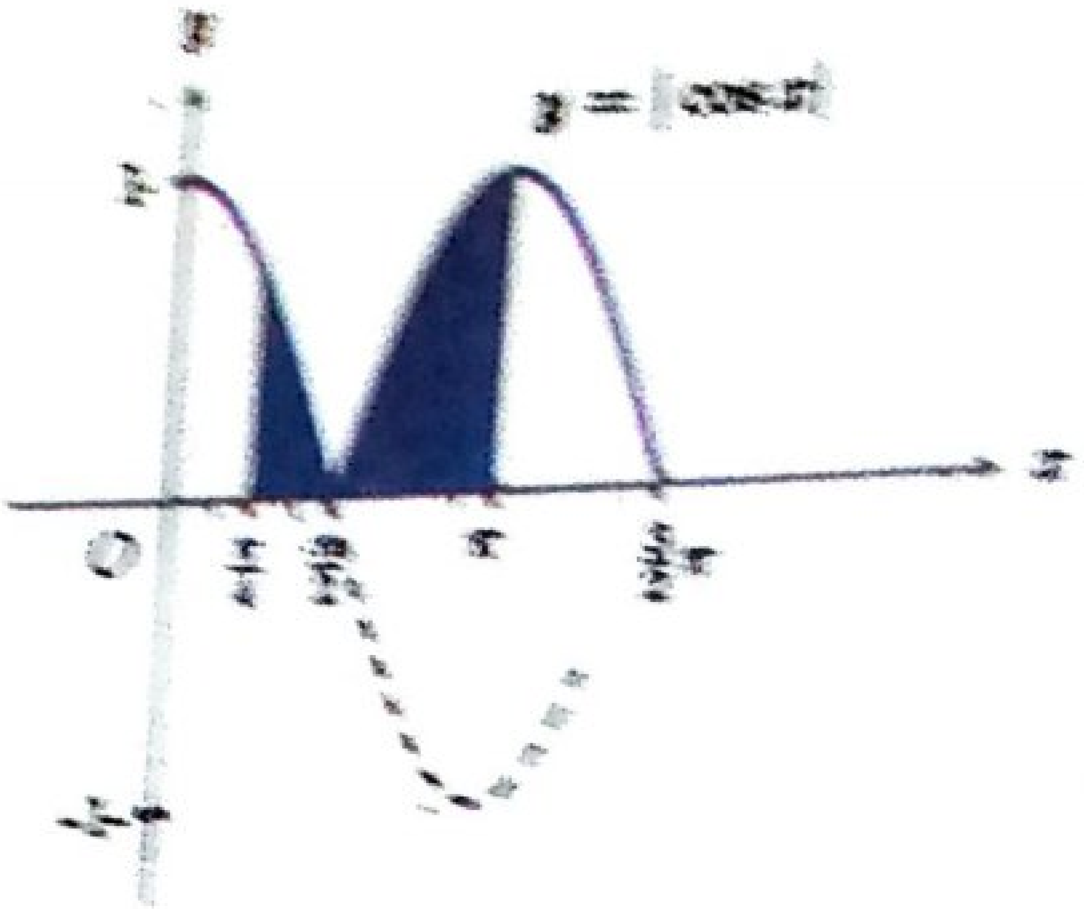
Solution

(a) $$\int_{-1}^{2} x^3\,dx = \frac{x^4}{4}\bigg]_{-1}^{2} = \frac{2^4}{4} - \frac{(-1)^4}{4} = 3\frac{3}{4}.$$

(b) $$\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\bigg]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1.$$

(c) $$\begin{aligned}\int_{-1}^{1} \sqrt{x+2}\,dx &= \int_{-1}^{1} (x+2)^{\frac{1}{2}}\,dx \\ &= \frac{2}{3}(x+2)^{\frac{3}{2}}\bigg]_{-1}^{1} \\ &= \frac{2}{3}\left((3)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right) \\ &= \frac{2}{3}(3\sqrt{3} - 1).\end{aligned}$$

d) $\int_{\frac{\pi}{4}}^{\pi} |\cos x| \; dx$



When $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$, $|\cos x| = \cos x$.

When $\frac{\pi}{2} \leq x \leq \pi$, $|\cos x| = -\cos x$.

$$\int_{\frac{\pi}{4}}^{\pi} |\cos x| \; dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \; dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \; dx$$

$$= \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \sin x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= (\sin\frac{\pi}{2} - \sin\frac{\pi}{4}) - (\sin\pi - \sin\frac{\pi}{2})$$

$$= 2 - \frac{\sqrt{2}}{2}.$$

Example 2.

Evaluate the integral $\int_1^2 x\sqrt{2-x}\,dx$.

Solution

Method 1

$$\int_1^2 x\sqrt{2-x}\,dx$$

Let $u = 2 - x$. Then $du = -dx$.

$$\int x\sqrt{2-x}\,dx = -\int (2-u)\sqrt{u}\,du$$

$$= -\int (2u^{\frac{1}{2}} - u^{\frac{3}{2}})du$$

$$= -\frac{4}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}$$

$$= -\frac{4}{3}(2-x)^{\frac{3}{2}} + \frac{2}{5}(2-x)^{\frac{5}{2}}.$$

Next

$$\int_1^2 x\sqrt{2-x}\,dx = \Big[-\frac{4}{3}(2-x)^{\frac{3}{2}} + \frac{2}{5}(2-x)^{\frac{5}{2}}\Big]_1^2$$
$$= 0 - (-\frac{4}{3} + \frac{2}{5}) = \frac{14}{15}.$$

Method 2

Let $u = 2 - x$. Then $du = -dx$.

Changing the variable from x to u, we get

x	$1 \to 2$
u	$1 \to 0$

$$\begin{aligned}\int_1^2 x\sqrt{2-x}\,dx &= -\int_1^0 (2u^{\frac{1}{2}} - u^{\frac{3}{2}})du\\ &= \int_0^1 (2u^{\frac{1}{2}} - u^{\frac{3}{2}})du\\ &= \Big[\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\Big]_0^1\\ &= \frac{14}{15}.\end{aligned}$$

Exercise 11.1

1. Evaluate the following definite integrals.

   (a) $\int_0^1 \frac{e^x}{e^x+1}\,dx$ (b) $\int_{-1}^1 x\sqrt{x+2}\,dx$ (c) $\int_0^6 (\frac{1}{3}x-1)^3\,dx$

   (d) $\int_2^4 \frac{x+5}{x^2-6x+5}\,dx$ (e) $\int_0^2 e^{-x}\cos 2x\,dx$ (f) $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \tan 2x\,dx$

2. Find the integral of

   $$\int_0^{2\pi} \cos mx \sin nx\,dx$$

   for (a) $m = n$ and (b) $m \neq n$.

3. Evaluate the following integrals.

   (a) $\int_0^4 |\sqrt{2x} - 2|\,dx$ (b) $\int_{-1}^1 |e^x - 1|\,dx$ (c) $\int_0^{2\pi} |\sin x|\,dx$

1.3 Area between the Curve and x-axis

this section, we discuss the area between the curve and x-axis using the definite egral. If $f(x)$ is positive and continuous on the closed interval $a \le x \le b$, then the a bounded by $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is given

$$\int_a^b f(x)\,dx.$$

s begin with the area enclosed between the graph and x-axis.

xample 3.

the area enclosed between the graph of

$f(x) = 3$ and the x-axis over $[1, 4]$.

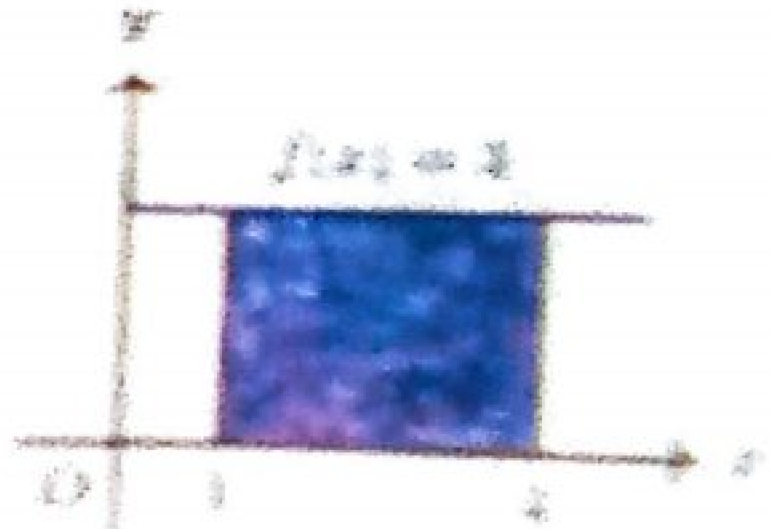
$f(x) = 3x + 2$ and the x-axis over $[-\frac{2}{3}, 0]$.

$f(x) = -x^2 + 4$ and the x-axis over $[-2, 2]$.

olution

$$\int_1^4 3\,dx = 3x\Big]_1^4$$

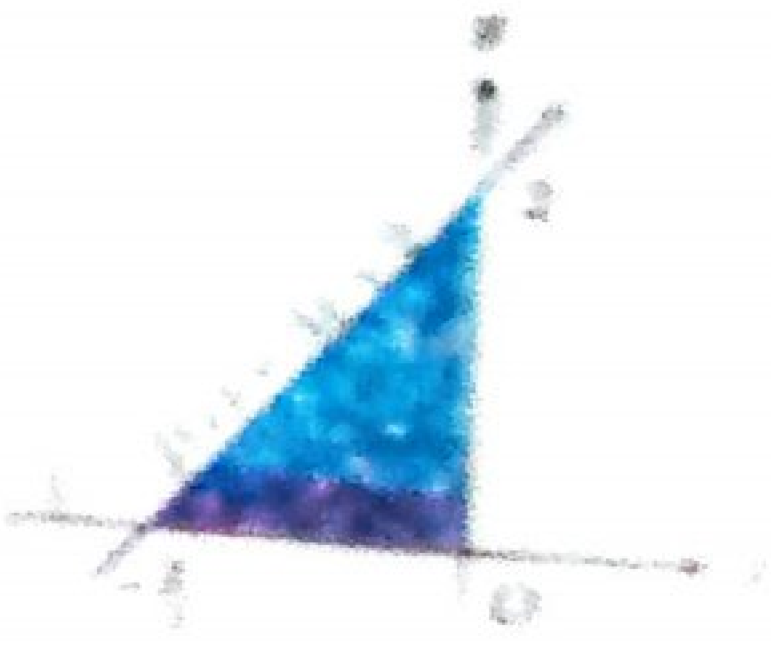
$$= 12 - 3 = 9 \text{ unit}^2.$$



$$\int_{-\frac{2}{3}}^{0} (3x + 2)\,dx = \left[3\,\frac{x^2}{2} + 2x\right]_{-\frac{2}{3}}^{0}$$

$$= 0 - \frac{3}{2}\left(-\frac{2}{3}\right)^2 - 2\left(-\frac{2}{3}\right)$$

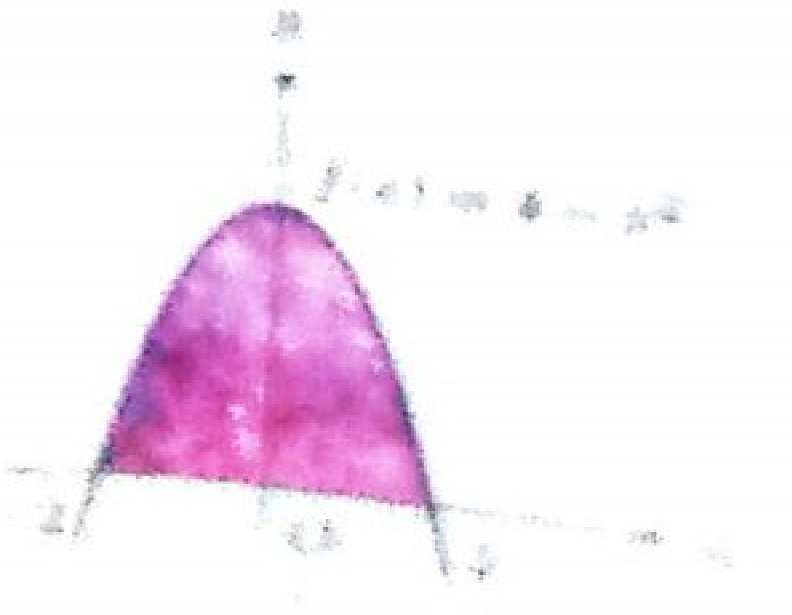
$$= \frac{2}{3} \text{ unit}^2.$$



$$\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 10\frac{2}{3} \text{ unit}^2.$$



Example 4.

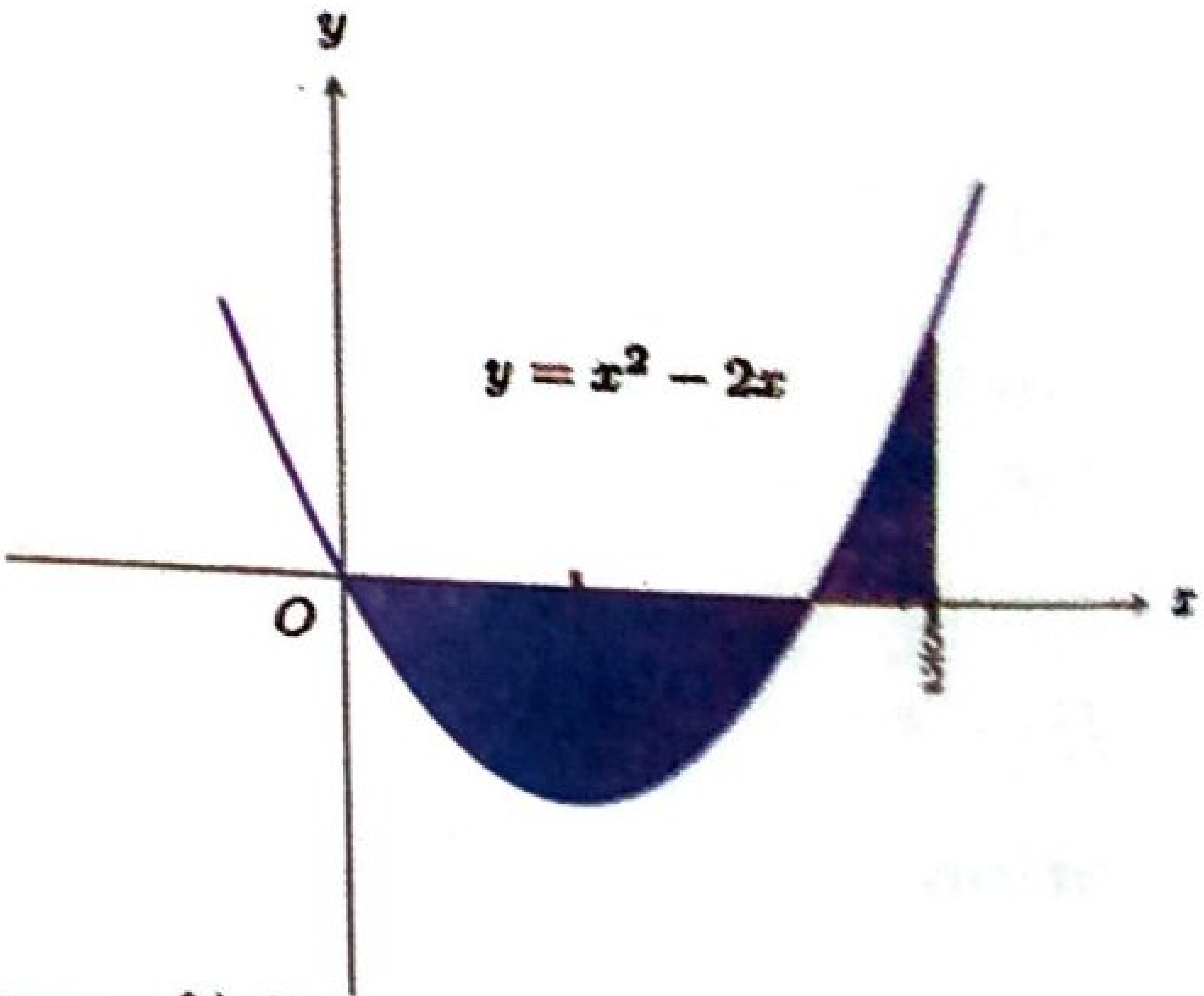
(a) Compute the definite integral $\int_0^{\frac{5}{2}} (x^2 - 2x)\, dx$.

(b) Find the total area enclosed between the graph of $f(x) = x^2 - 2x$ and the x-axis over $[0, \frac{5}{2}]$.

Solution

(a)

$$\begin{aligned}
&\int_0^{\frac{5}{2}} (x^2 - 2x)\, dx \\
&= \left[\frac{x^3}{3} - x^2\right]_0^{\frac{5}{2}} \\
&= \frac{1}{3}\left(\frac{5}{2}\right)^3 - \left(\frac{5}{2}\right)^2 \\
&= -1\frac{1}{24}.
\end{aligned}$$



(b) For the intersection point of x-axis and the curve, $f(x) = 0$

$$\begin{aligned}
x^2 - 2x &= 0 \\
x(x-2) &= 0 \\
x &= 0,\ 2.
\end{aligned}$$

Thus the graph of $f(x)$ cuts x-axis at 0 and 2, we divide the domain in two intervals: the interval $[0, 2]$ and the interval $[2, \frac{5}{2}]$.

$$\int_0^2 (x^2 - 2x)\, dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - 0 = -\frac{4}{3}.$$

On $[0, 2]$, the function $f \leq 0$, so the value of integration f gives negative.

$$\begin{aligned}
\int_2^{\frac{5}{2}} (x^2 - 2x)\, dx &= \left[\frac{x^3}{3} - x^2\right]_2^{\frac{5}{2}} \\
&= \left(\frac{125}{24} - \frac{25}{4}\right) - \left(\frac{8}{3} - 4\right) \\
&= \frac{7}{24}.
\end{aligned}$$

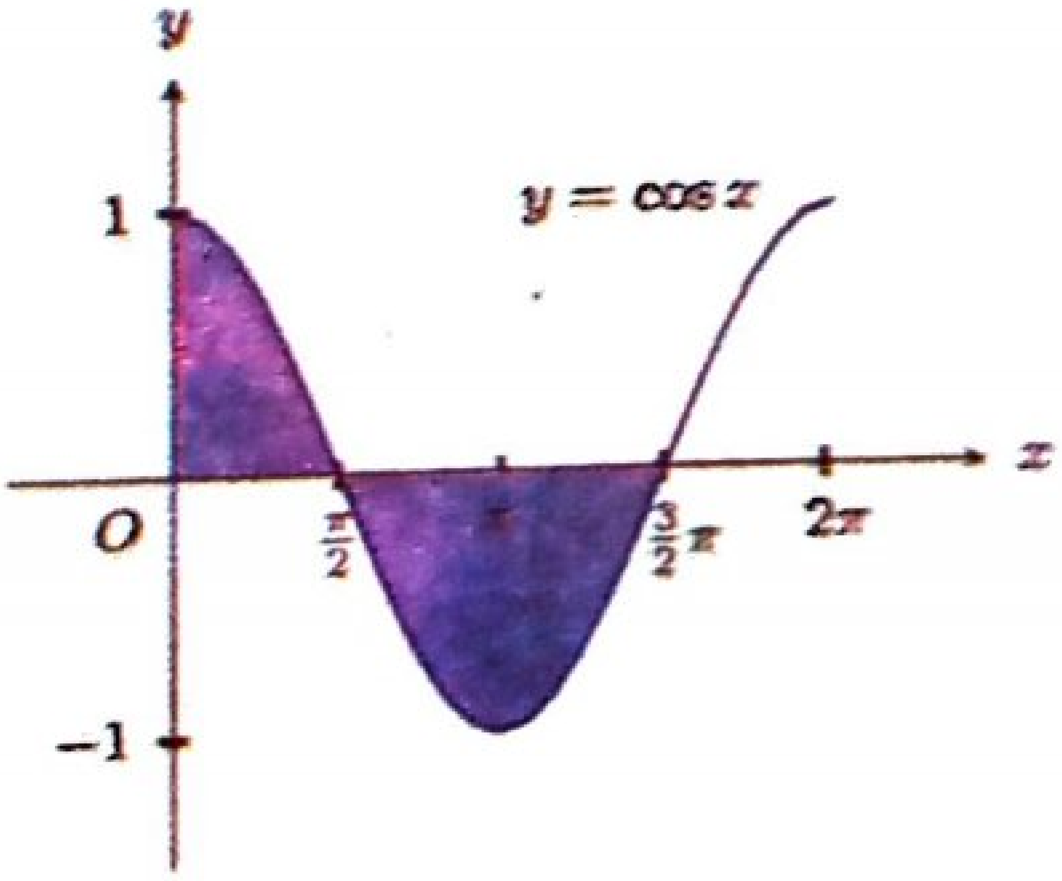
On $[2, \frac{5}{2}]$, the function $f \geq 0$, so the value of integration f gives positive.

Therefore,

$$\text{Area} = |-\frac{4}{3}| + |\frac{7}{24}| = 1\frac{5}{8} \text{ unit}^2.$$

Example 5.

Figure shows the graph of the function $f(x) = \cos x$.



Find the area enclosed between the graph of $f(x) = \cos x$ and the lines

(a) $x = 0$ and $x = \frac{\pi}{2}$ (b) $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ (c) $x = 0$ and $x = \frac{3\pi}{2}$.

(d) Find the definite integral $\int_0^{\frac{3\pi}{2}} \cos x \, dx$.

Solution

(a) $$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

Area = 1 unit^2.

(b) $$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx = \sin x \Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} = -2.$$

Area $= |-2| = 2$ unit^2.

(c) Area between the graph $f(x) = \cos x$ and the lines $x = 0$ and $x = \frac{3\pi}{2}$ is 3 unit^2.

(d) $$\int_0^{\frac{3\pi}{2}} \cos x \, dx = \sin x \Big]_0^{\frac{3\pi}{2}} = \sin\frac{3\pi}{2} - \sin 0 = -1.$$

To find the area between the graph of $y = f(x)$ and x-axis over the interval $[a, b]$:

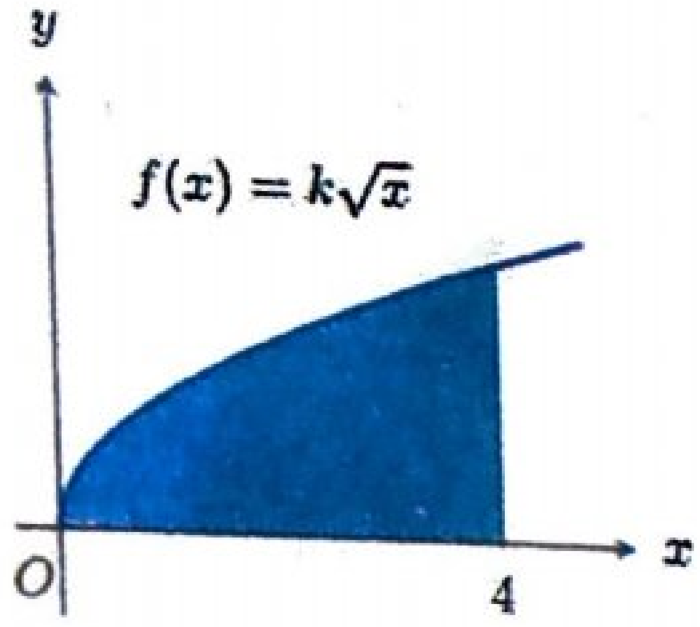
Step 1 Find the intersection point(s) of the graph $y = f(x)$ and x-axis.

Step 2 Divide the interval $[a, b]$ at the zeros of f.

Step 3 Integrate f over each subinterval.

Step 4 Add the absolute value of the integrals.

Example 6.



The shaded area is 16 unit^2. Find the value of k.

Solution

$$\int_0^4 k\sqrt{x}\,dx = \frac{2k}{3}x^{\frac{3}{2}}\Big]_0^4 = \frac{2k}{3}\cdot 4^{\frac{3}{2}} = \frac{16k}{3}.$$

By the problem, $\dfrac{16k}{3} = 16$, so $k = 3$.

.4 Area between Two Curves

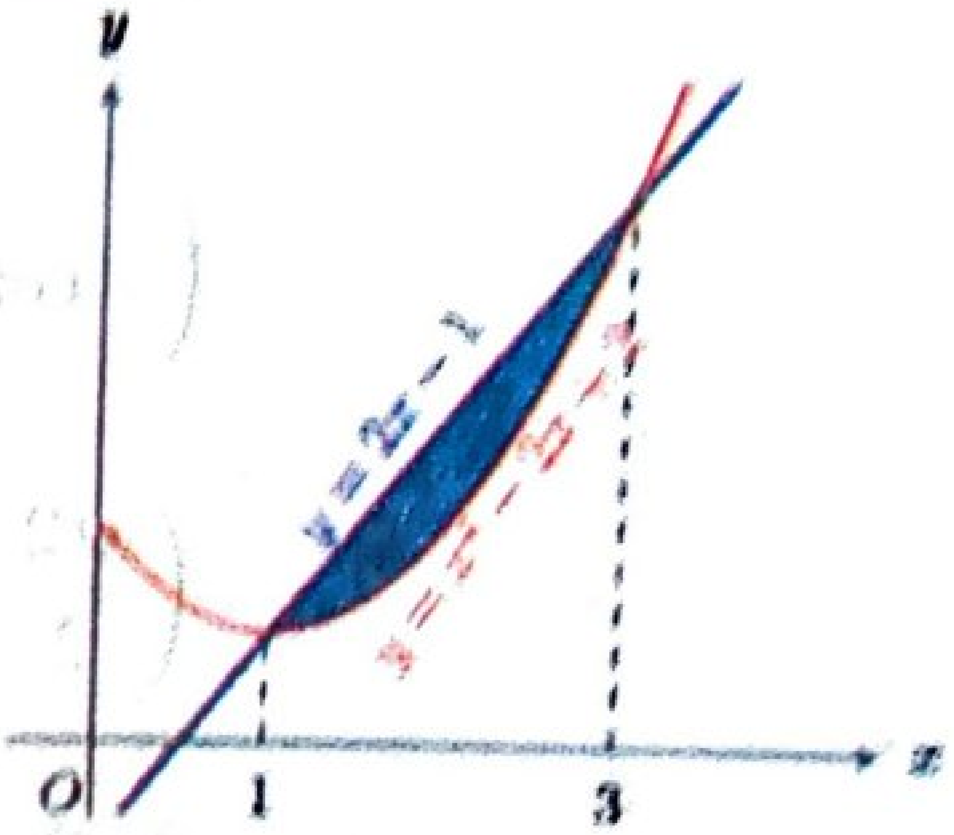
this section we consider the areas bounded by two curves.

ample 7.

nd the area between $y = x^2 - 2x + 2$ and $y = 2x - 1$.

olution

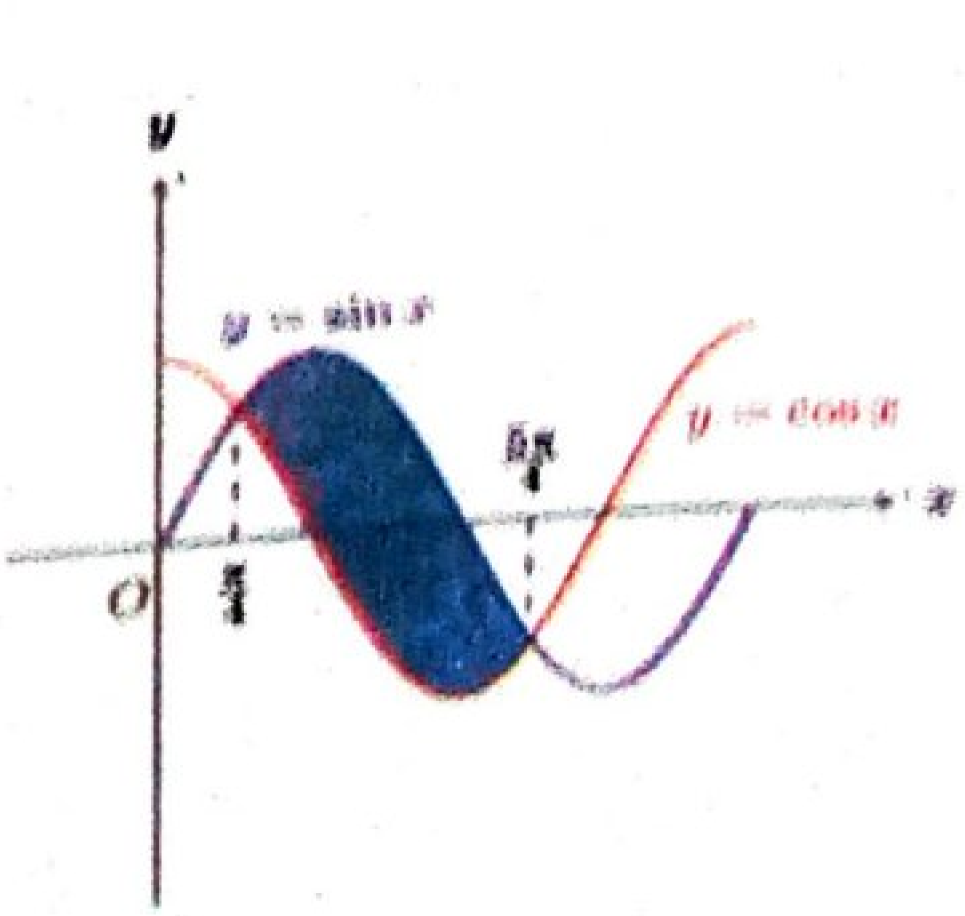
find the x-coordinate of intersection ints,



$$\begin{aligned} x^2 - 2x + 2 &= 2x - 1 \\ x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x &= 1, 3. \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_1^3 [(2x-1) - (x^2 - 2x + 2)]dx \\ &= \int_1^3 (-x^2 + 4x - 3)\, dx \\ &= \left[-\frac{x^3}{3} + \frac{4x^2}{2} - 3x\right]_1^3 = 0 - (-\frac{4}{3}) = 1\frac{1}{3}. \end{aligned}$$

Example 8.



Find the area between $y = \sin x$ and $y = \cos x$ for $\frac{\pi}{4} \leq x \leq \frac{5}{4}\pi$.

Solution

From the figure, we see that $\sin x \geq \cos x$ when $\frac{\pi}{4} \leq x \leq \frac{5}{4}\pi$.

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx$$

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2}.$$

To find the area between the graph of $y = f(x)$ and $y = g(x)$ over the interval $[a, b]$:

Step 1 Find the intersection points of the graphs $y = f(x)$ and $g(x)$.

Step 2 Divide the interval $[a, b]$ at the intersection points.

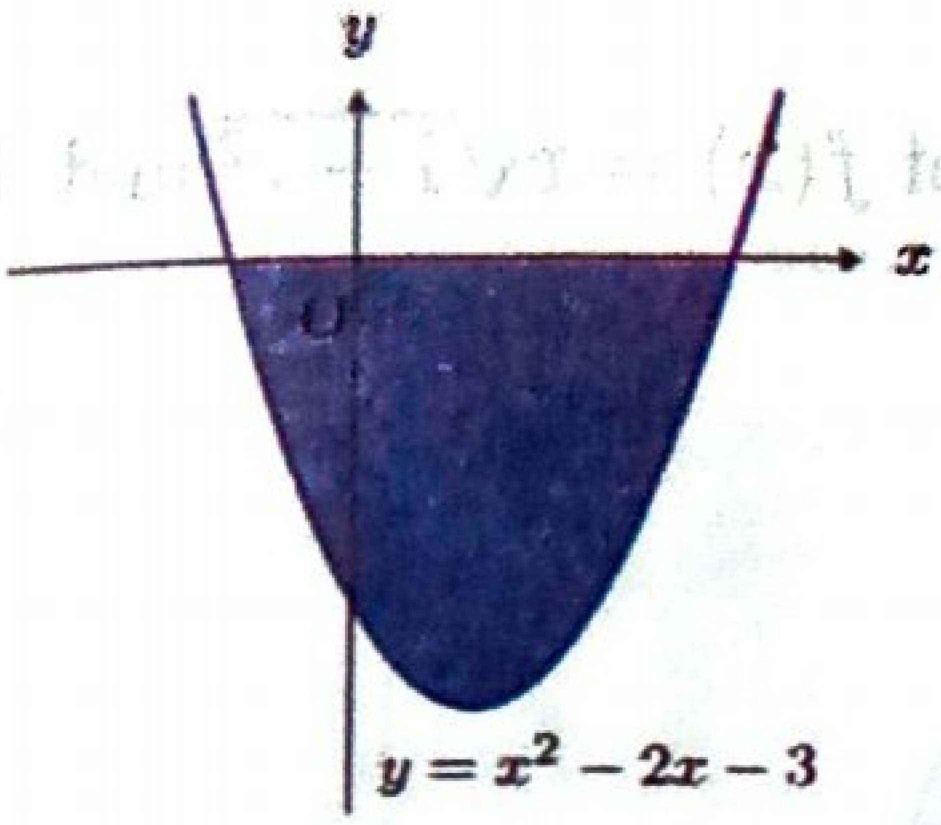
Step 3 Investigate which function is on the top at each subinterval.

Step 4 Integrate $(f - g)$ if f is on the top, or $(g - f)$ if g is on the top over each subinterval.

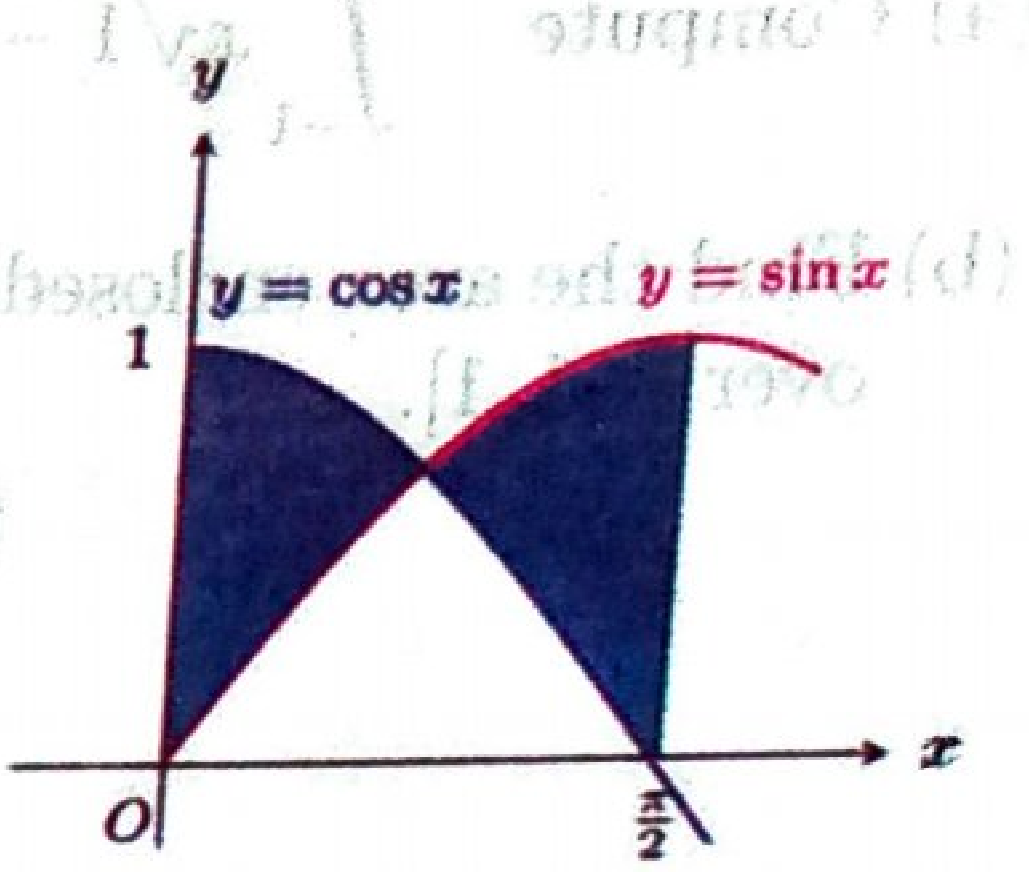
Exercise 11.2

1. Find the shaded areas.

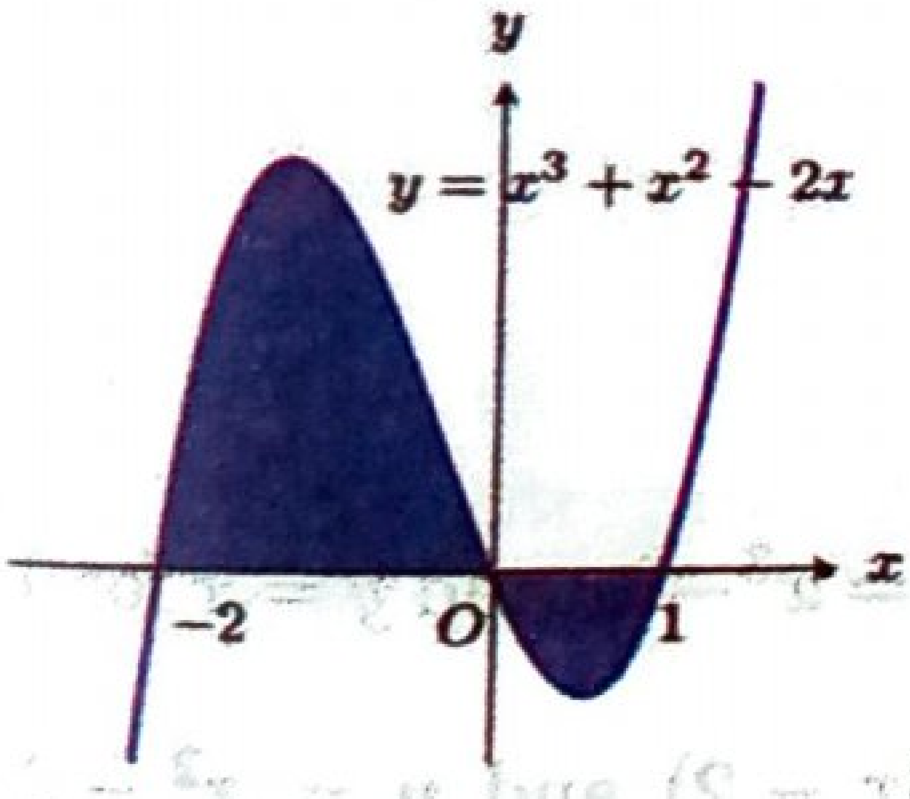
(a)



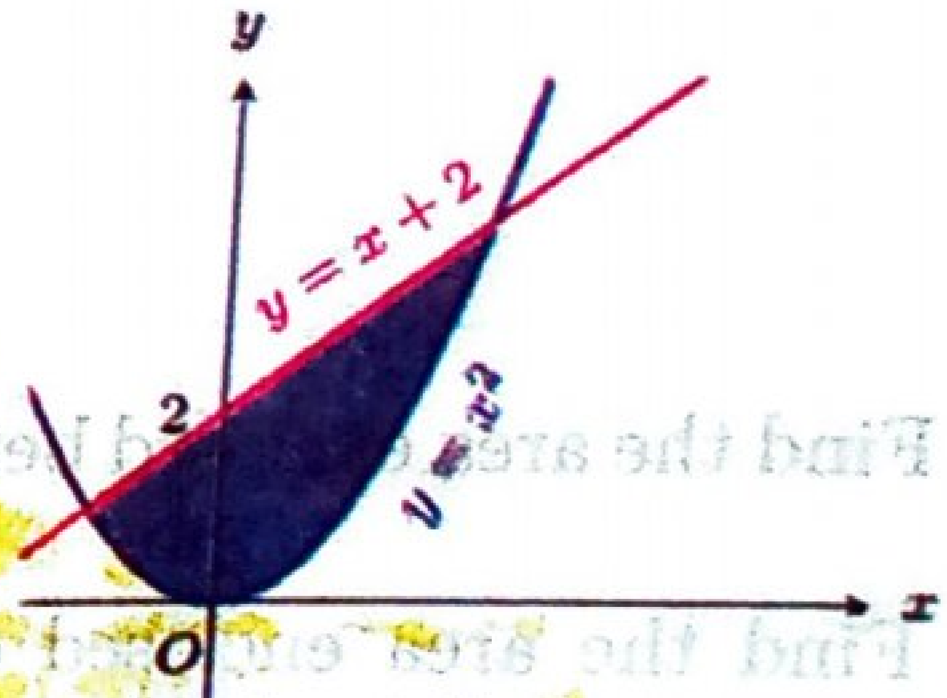
(b)



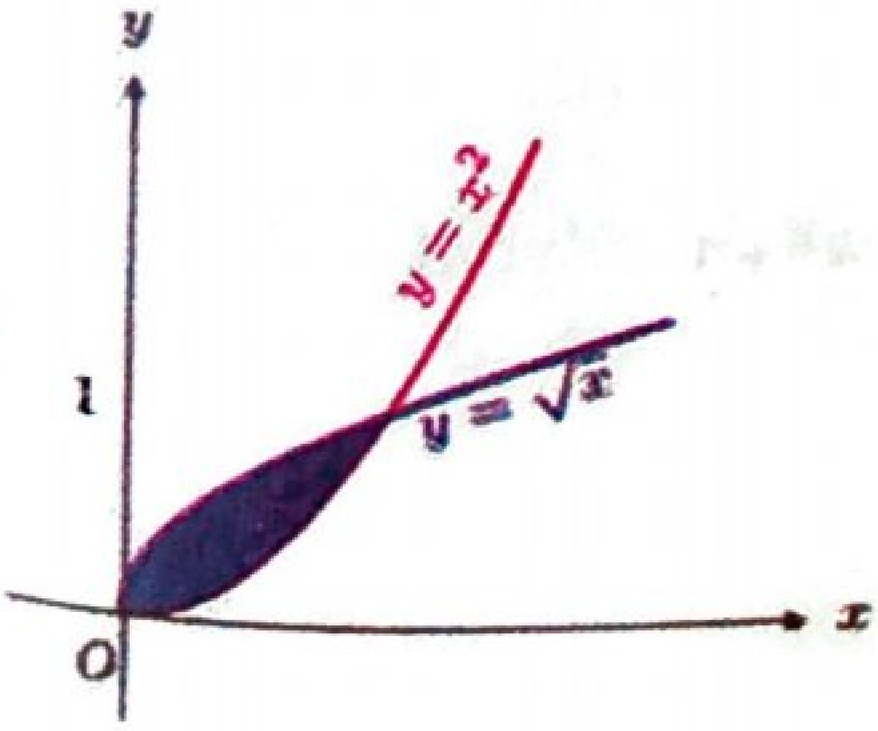
(c)



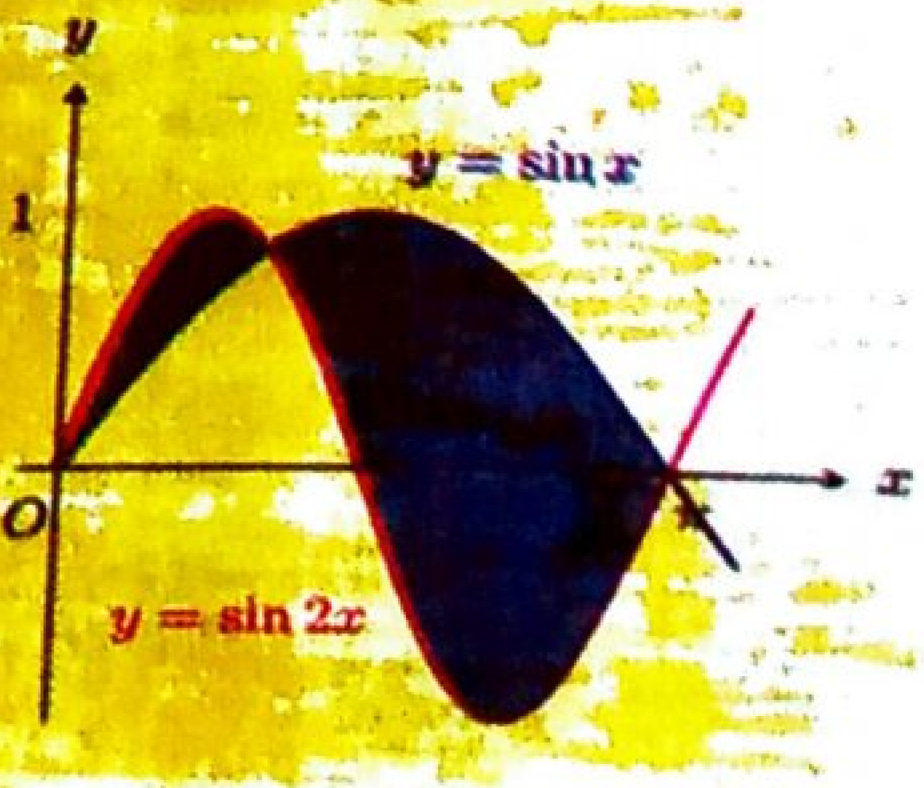
(d)



(e)



(f)



2. Find the area under each curve between the given x-values:

(a) $f(x) = \ln x$ from $x = 1$ to $x = e$.

(b) $f(x) = 12 - 3x^2$ from $x = -1$ to $x = 2$.

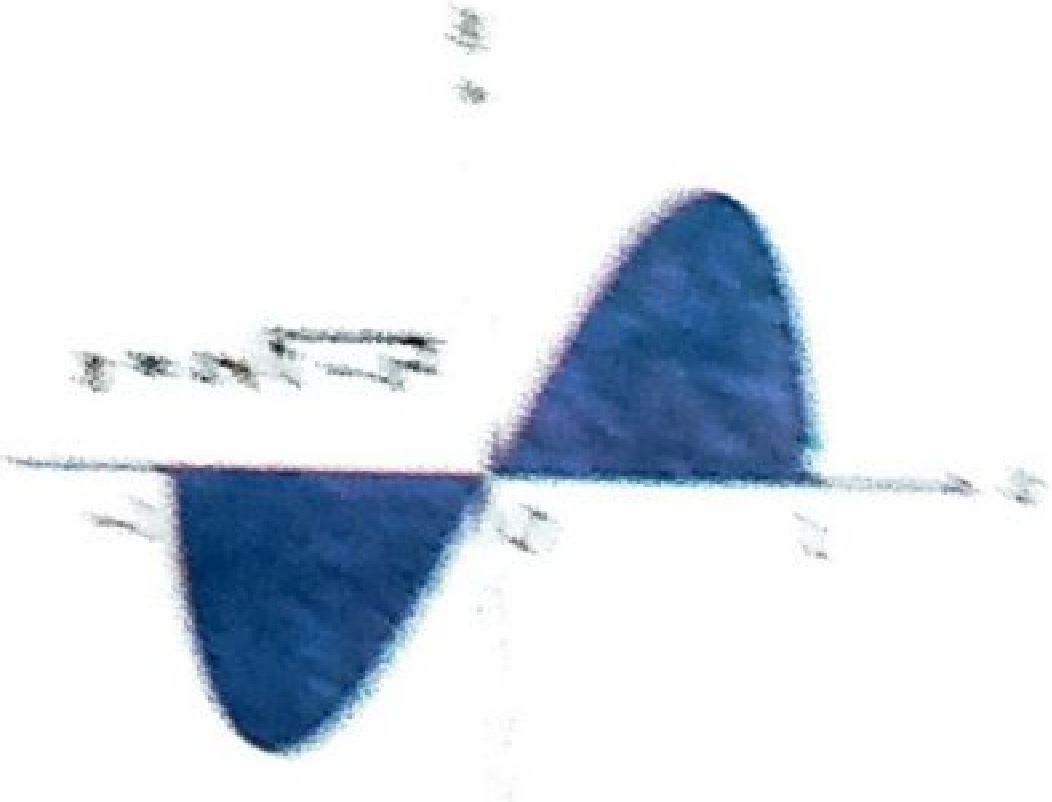
(c) $f(x) = xe^{x^2}$ from $x = 0$ to $x = 1$.

(d) $f(x) = \sin x$ from $x = 0$ to $x = \frac{3\pi}{2}$.

(e) $f(x) = (1 + \cos x)\sin x$ from $x = 0$ to $x = \pi$.

3. (a) Compute $\int_{-1}^{1} x\sqrt{1-x^2}\,dx$.

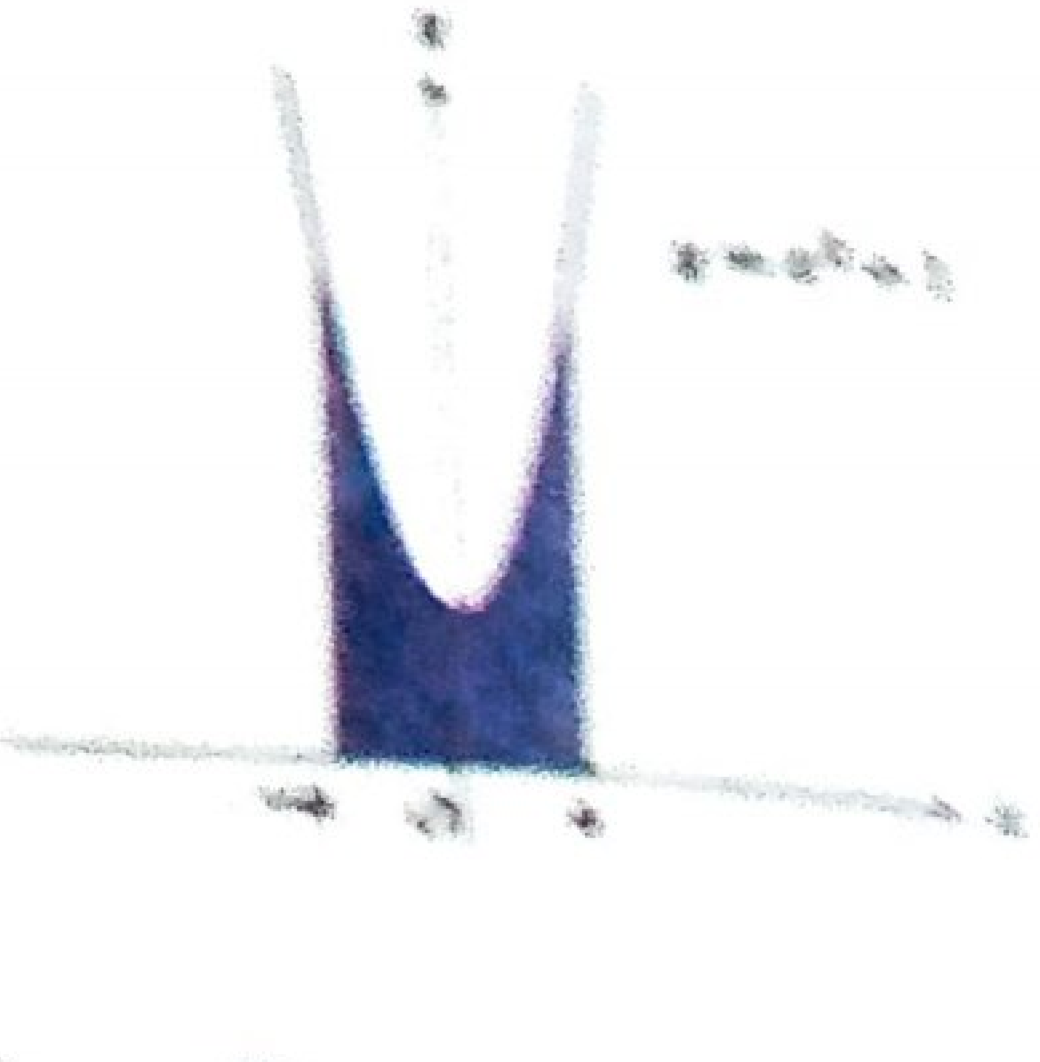
   (b) Find the area enclosed between the graph of $f(x) = x\sqrt{1-x^2}$ and the x-axis over $[-1, 1]$.



4. Find the area enclosed between the graphs of $y = x^2 - 3x$ and $y = x$ for $0 \le x \le 2$.

5. Find the area enclosed of the graphs of $\ln(3x - 2)$ and $y = x^2 - 2x + 1$ for $1 \le x \le 2$.

6. The shaded area is $8a$ unit^2. Find the value of a.

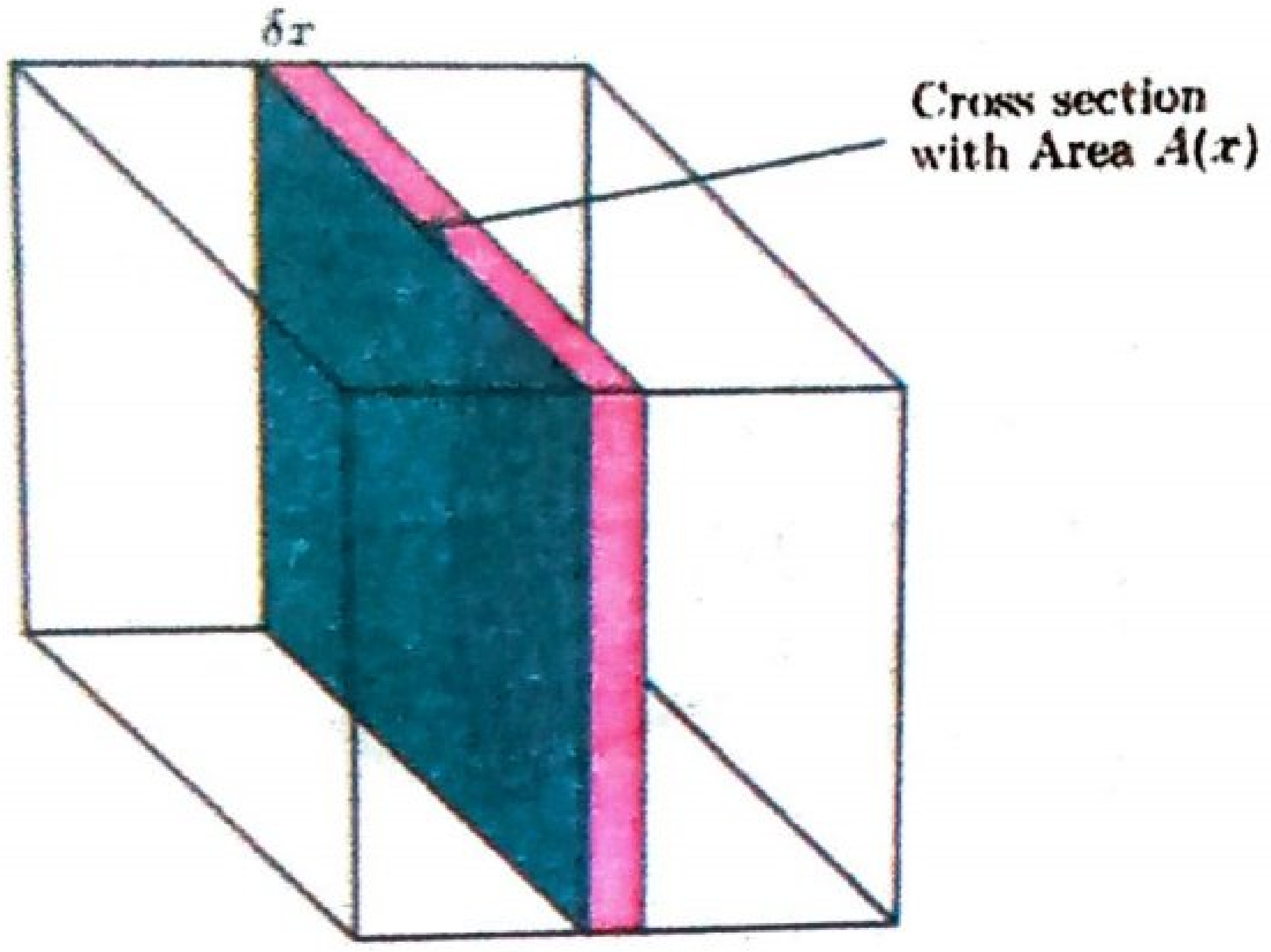


11.5 Volumes Using Cross-Sections

We studied the area between a curve and x-axis, and two curves. We now consider the volume of a particular solid by integration: the method of slicing, the disc method and the washer method.

Volume by Method of Slicing

In this section, we define volumes of solids using the areas of their cross-section. A **cross-section** of a solid is the plane region generated by intersecting the solid with a plane.

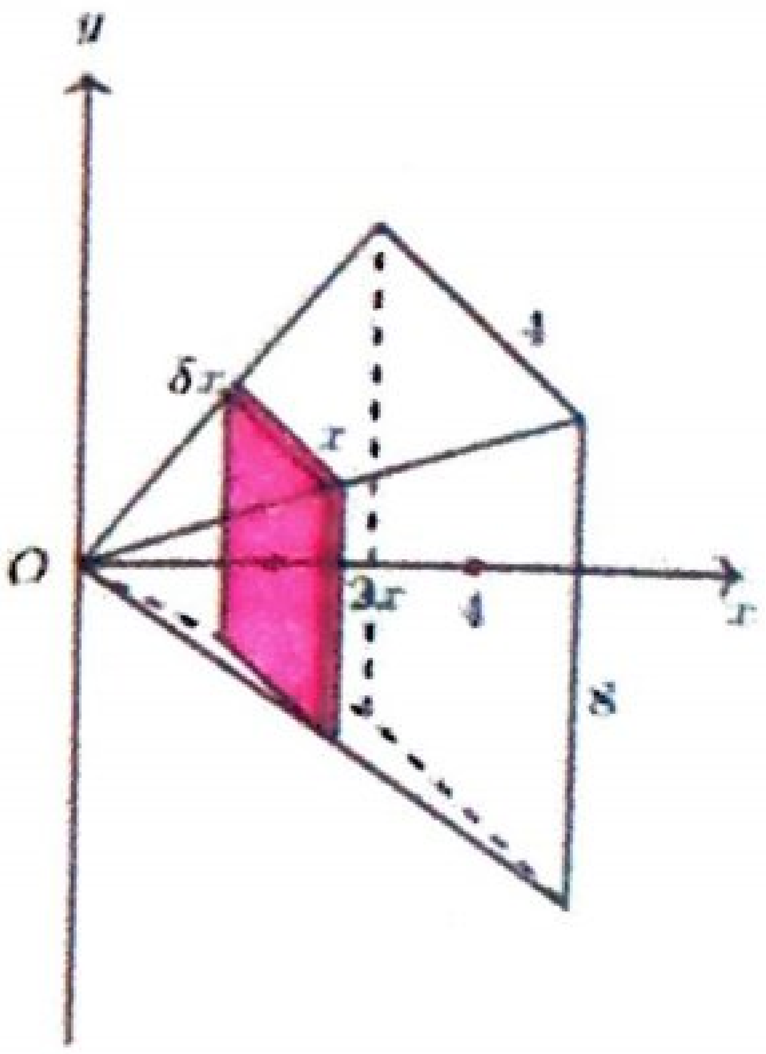


Suppose $A(x)$ is the integrable cross-section area from $x = a$ to $x = b$. We define the volume of the solid as

$$\boxed{\textbf{Volume} = \int_a^b A(x)\,dx.}$$

Example 9.

A pyramid with the height of 4 m has a rectangle base shown in figure. Find the volume of the pyramid.



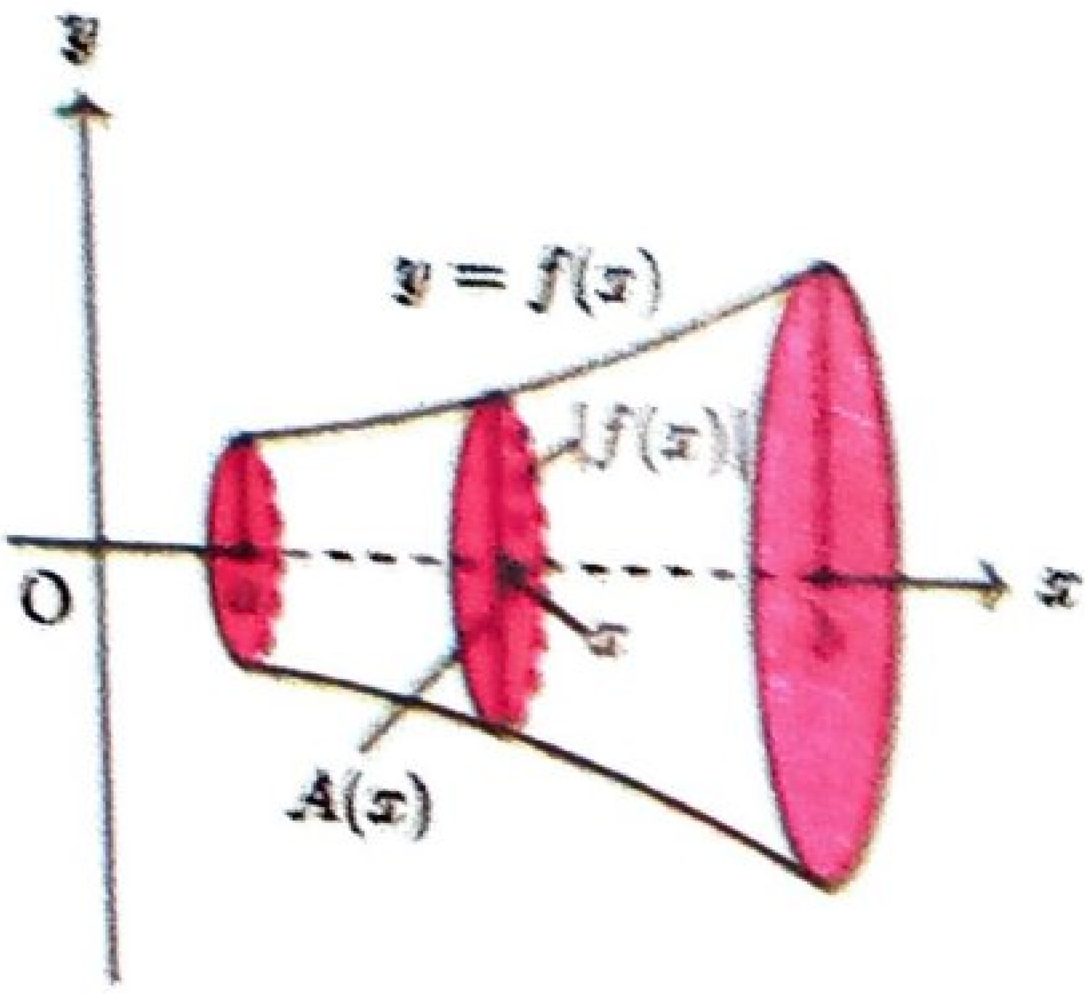
Solution

The cross-section area at x is $A(x) = 2x^2$.

$$\begin{aligned} V = \int_0^4 A(x)\,dx &= \int_0^4 2x^2\,dx \\ &= \frac{2x^3}{3}\Big]_0^4 \\ &= \frac{128}{3}\ \text{m}^3. \end{aligned}$$

Volume of Revolution (Disc Method)

Consider the curve $y = f(x)$ for $a \leq x \leq b$. If the shaded part is rotated about the x-axis through 360°, a three-dimensional solid will be formed. This solid is called **a solid of revolution**. We can find the volume of this generated solid by definite integration.



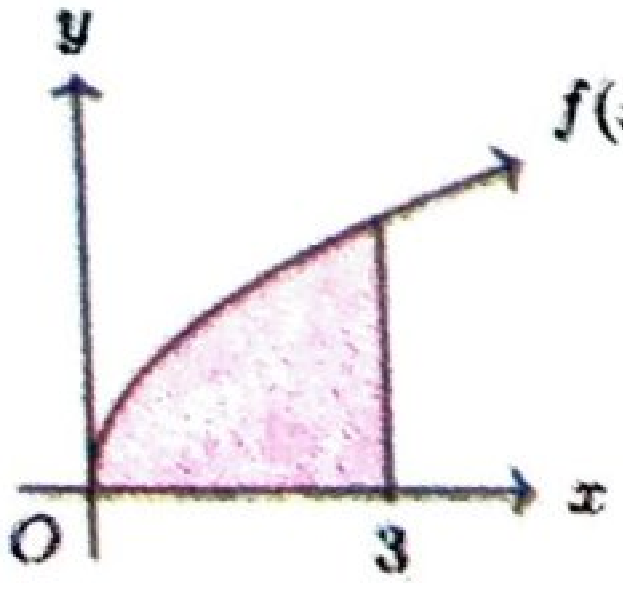
$$\textbf{Volume of revolution} = \pi \int_a^b [f(x)]^2 \, dx.$$

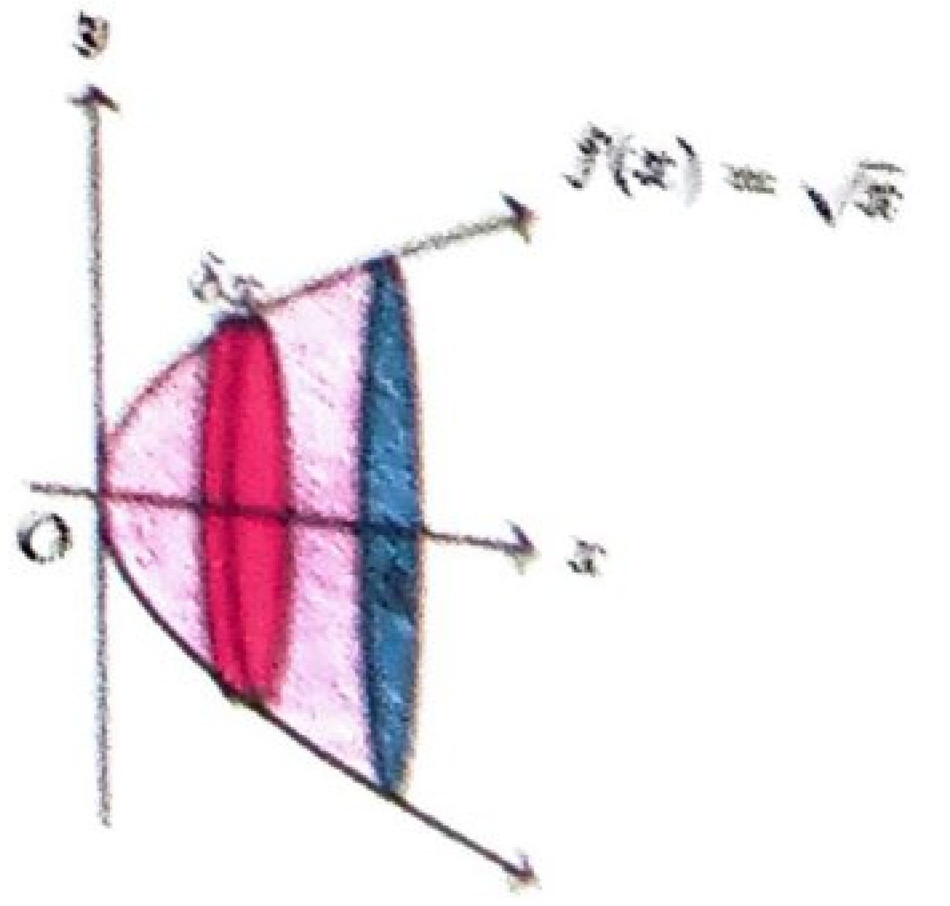
Example 10.

Find the volume of revolution formed when the curve
(a) $y = \sqrt{x}$ for $0 < x \leq 3$
(b) $y = x^2$ for $0 < x < 1$
are rotated through 360° about the x-axis.

Solution

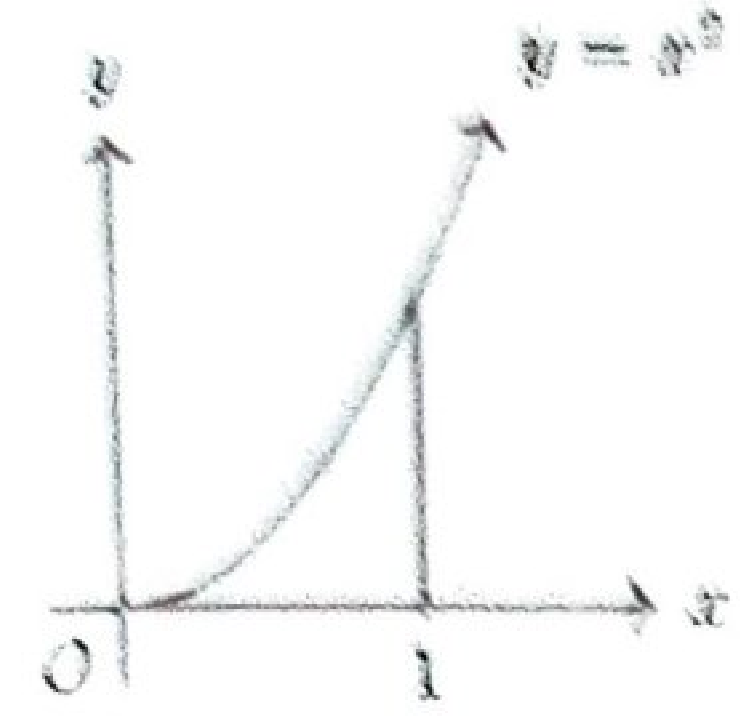
(a)

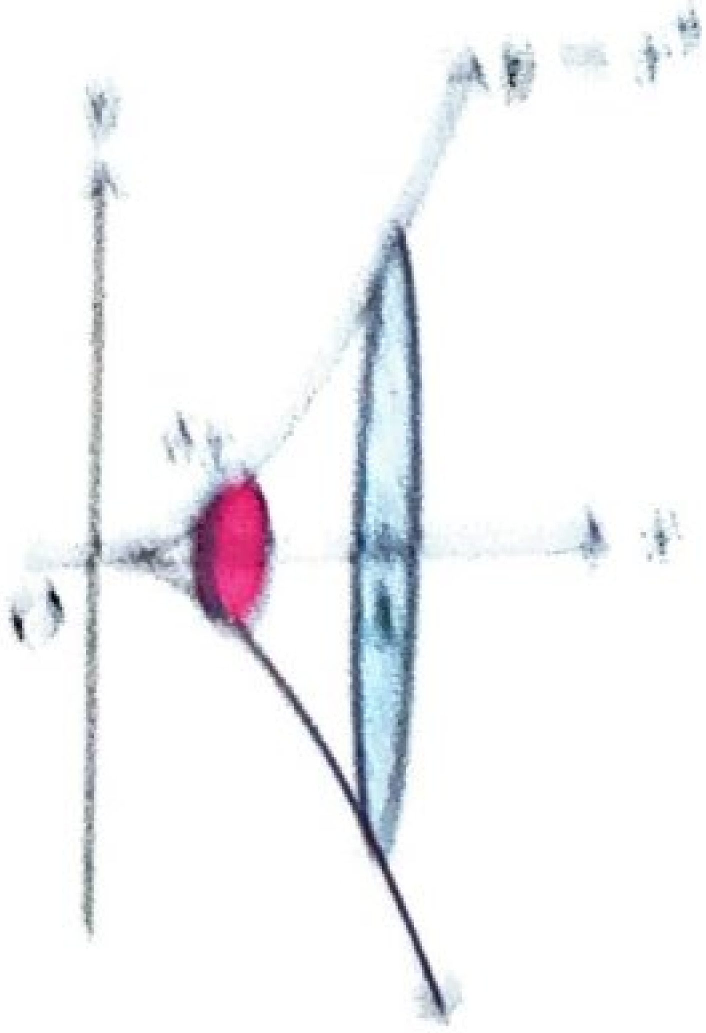




$$\text{Volume of revolution} = \pi \int_0^3 (\sqrt{x})^2 \, dx = \pi \int_0^3 x \, dx$$
$$= \pi \frac{x^2}{2}\Big]_0^3 = \frac{9\pi}{2} \text{ unit}^3$$

(b)



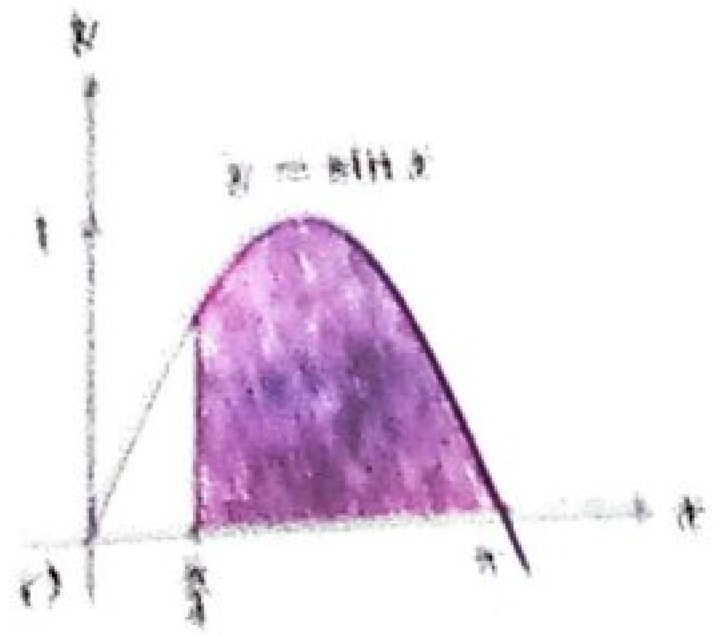


$$\text{Volume of revolution} = \pi\int_0^1 (x^2)^2\,dx = \pi\int_0^1 x^4\,dx$$
$$= \pi\,\frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}\text{ unit}^3$$

Example 11.

Find the volume of revolution formed when the curve $y = \sin x$ for $\frac{\pi}{4} \le x \le \pi$, is rotated through 360° about the x-axis.

Solution



$$\text{Volume of revolution} = \pi\int_{\frac{\pi}{4}}^{\pi} \sin^2 x\,dx = \pi\int_{\frac{\pi}{4}}^{\pi} \frac{1-\cos 2x}{2}\,dx$$
$$= \pi\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\frac{\pi}{4}}^{\pi}$$
$$= \pi\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \pi\left(\frac{\pi}{8} - \frac{\sin\frac{\pi}{2}}{4}\right)$$
$$= \pi\left(\frac{\pi}{2} - 0\right) - \pi\left(\frac{\pi}{8} - \frac{1}{4}\right)$$
$$= \left(\frac{3\pi^2}{8} + \frac{\pi}{4}\right)\text{ unit}^3$$

olume of Revolution (Washer Method)

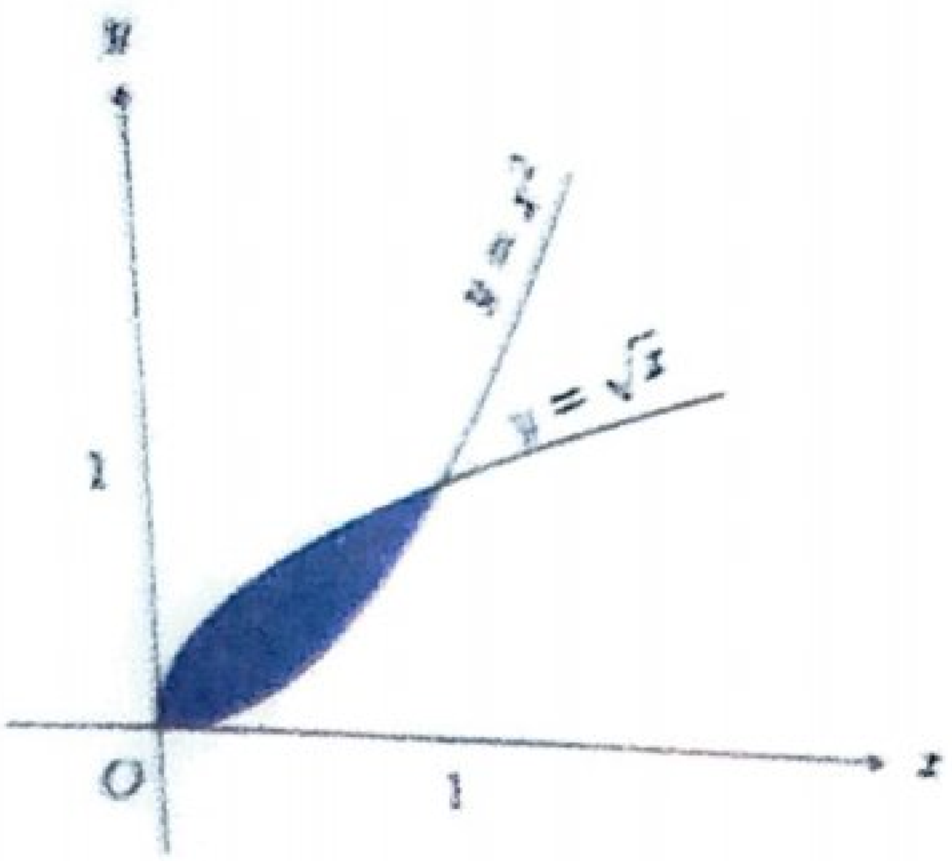
the region bounded by $y = f(x)$ (on top) and $y = g(x)$ and the limits $x = a$, $x = b$ revolved about the x-axis, then its volume of revolution is given by

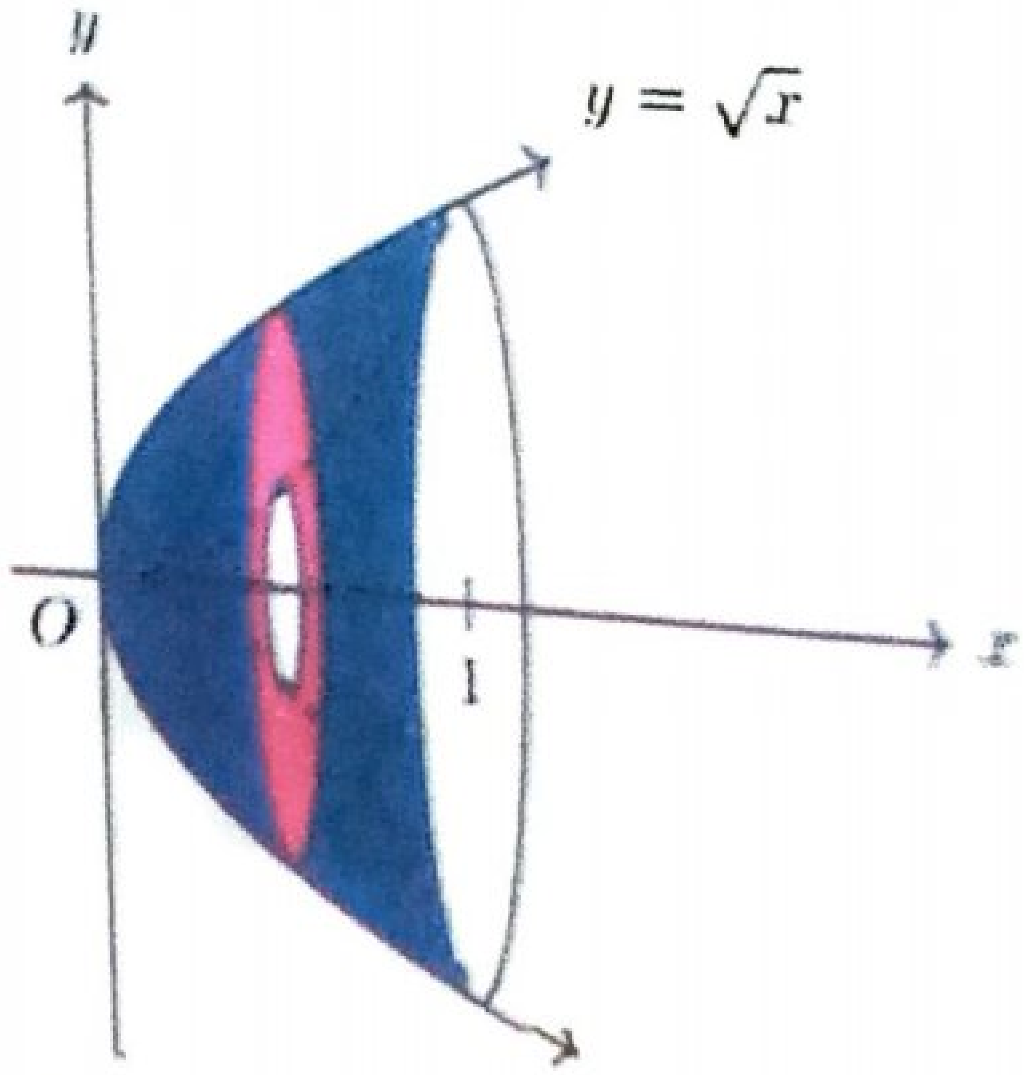
$$\textbf{Volume of revolution} = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right]\, dx.$$

xample 12.

ind the volume of revolution generated by revolving the regions between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.

Solution





The two graphs cut x-axis at 0 and 1. The curve $y = \sqrt{x}$ is bounded above.

$$\begin{aligned} V &= \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right]\, dx \\ &= \pi \int_0^1 \left[x - x^4\right] dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10} \text{ unit}^3. \end{aligned}$$

To find the volume of revolution formed the graph of $y = f(x)$, $y = g(x)$ rotated about x-axis:

Step 1 Find the intersection points of the graphs $y = f(x)$ and $g(x)$.

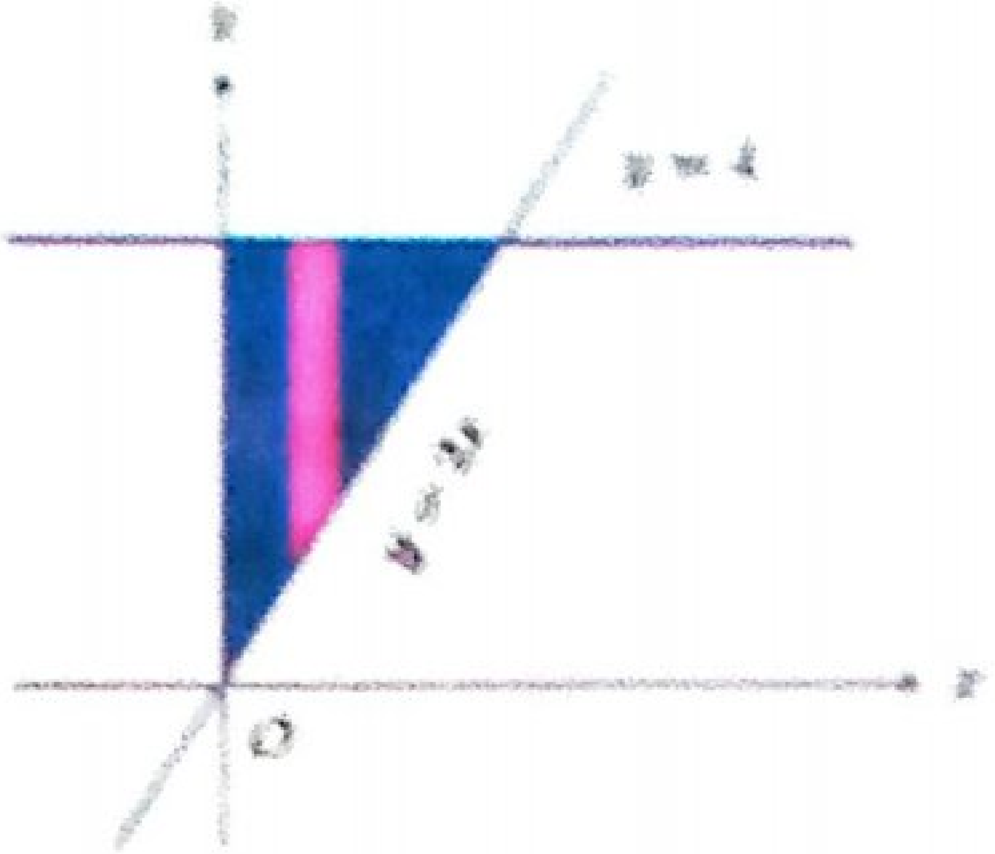
Step 2 Investigate which function is on the top.

Step 3 Integrate $(f^2 - g^2)$ if f is on the top, or $(g^2 - f^2)$ if g is on the top on the interval.

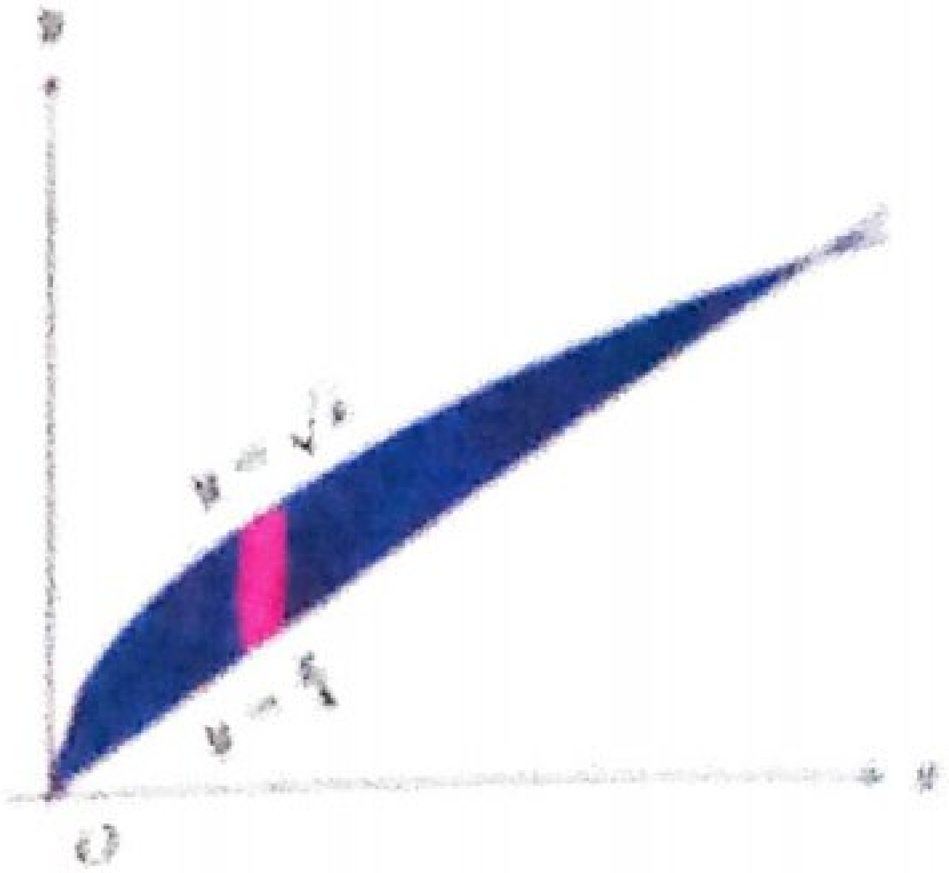
Exercise 11.3

the volume of solid by using slicing method in Exercise 1-3.

1. The base of the solid is the region bounded by the graphs of $y = 2x$, $y = 4$ and $x = 0$. The cross-section perpendicular to the x-axis are
   (a) rectangle of height 8.
   (b) rectangle of perimeter 10.



2. The base of the solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{4}$. The cross-section perpendicular to the x-axis are
   (a) isosceles triangles of height 4.
   (b) semicircles with diameters running across the base of solid.



The solid lies between the planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-section perpendicular to the x-axis are circular disks whose diameters run from $y = -x^2 + 4$ and $x^2 + 2$.

4. By using the disc method, find the volume of revolution formed when the curve $y = f(x)$ for $a \le x \le b$, is rotated through 360° about the x-axis.

(a) $f(x) = 3x + 2;\quad a = \frac{1}{2},\quad b = 4.$

(b) $f(x) = \sqrt{\cos x};\quad a = 0,\quad b = \frac{\pi}{2}.$

(c) $f(x) = \frac{1}{x-1};\quad a = 3,\quad b = 5.$

(d) $f(x) = \sin x \cos x;\quad a = 0,\quad b = \frac{\pi}{2}.$

(e) $f(x) = \sqrt{x}\,\sqrt[4]{1+x^2};\quad a = 2,\quad b = 3.$

5. By using the washer method, find the volume of revolution generated by revolving the regions between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ about the x-axis.

(a) $f(x) = x^2,\quad g(x) = 2x;\quad a = 0,\quad b = 2.$

(b) $f(x) = \tan x,\quad g(x) = \frac{x}{2};\quad a = 0,\quad b = \frac{\pi}{4}.$

(c) $f(x) = 2\sqrt{x},\quad g(x) = 2;\quad a = 0,\quad b = 1.$